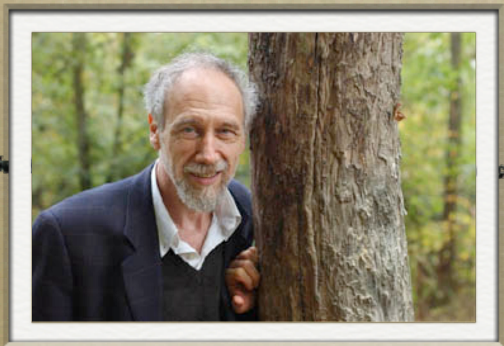
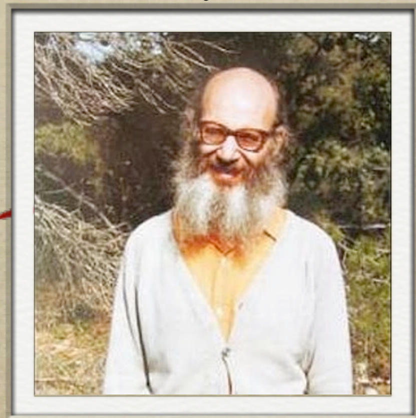
The Gelfand spectrum of a noncommutative C*-algebra:

from noncommutative geometry to topos theory





Klaas Landsman

Radboud Universiteit Nijmegen

B. Banaschewski & C.J. Mulvey, A globalisation of the Gelfand duality theorem
Annals of Pure and Applied Logic 137, 62-103 (2006)

C. Heunen, N.P. Landsman, B. Spitters, A topos for algebraic quantum theory
Communications in Mathematical Physics 291, 63-110 (2009)

Reminder: Gelfand duality

- *Compact space X ☞ C(X) ≡ C(X, ℂ) as C*-algebra*
- *cpt Hausdorff spaces ≃ (unital commutative C*-algebras)^{op}*
- ***quantum jump:** "noncommutative spaces" ≃ (C*-algebras)^{op}*
- ***Amazing fact:** by Gelfand-Naimark Theorem, noncommutative spaces relate to **Hilbert space***

  ☞ Noncommutative spin manifolds ≃ ("spectral triples")^{op}

  Are there other ways to capture spaces algebraically?

Order-theoretic approach

- *Space X ☞ topology O(X) as* ***lattice***, $U \leq W$ *iff* $U \subseteq W$
- *Fine structure: O(X) is special lattice called* ***frame***,
  i.e., complete lattice such that $U \wedge \bigvee_i \{W_i\} = \bigvee_i \{U \wedge W_i\}$
- ***Point*** *of frame F is frame map* $p: F \to \underline{2} \equiv \{0,1\} = O(\circledast)$
- ***Points***(*F*) *topologized by opens* $\{p \mid p(U)=1\}$, $U \in F$
- *Frame F is called* ***spatial*** *if* $F \cong O(\textbf{Points}(F))$
- *Space X is called* ***sober*** *if* $X \cong \textbf{Points}(O(X))$

➡ ***"Stone" duality***: *Sober spaces* $\simeq$ *(spatial frames)*^{op}

Pointfree spaces and logic

- ***Leap of faith***: *"pointfree spaces" ≃ (frames)^{op}*
- ***Surprising fact***: *pointfree spaces (locales) relate to* ***logic***
- ***Heyting algebra*** *is lattice H with top ⊤, bottom ⊥, and map $\Rightarrow: H \to H$ such that $A \leq (B \Rightarrow C)$ iff $(A \wedge B) \leq C$*
- *Heyting algebras describe* ***intuitionistic propositional logic,*** *with* ***negation*** *$\neg A := (A \Rightarrow \bot)$: typically $A \vee \neg A \neq \top$, $\neg\neg A \neq A$*
- ***Frame ↔ complete Heyting algebra***: *$(B \Rightarrow C) = \bigvee\{A \mid (A \wedge B) \leq C\}$*
- ***So***: *spaces ☞ spatial frames ☞ pointfree spaces ☞ logic*

  Cf: spaces ☞ comm. C-algebra's ☞ noncommutative spaces ☞ Hilbert space*

Constructive Gelfand duality

- *Gelfand duality $A \cong C(X)$ not valid constructively*
  (no problem in set theory, but problematic in topos theory)
- *For sober spaces one has $C(X,Y) \cong$ **Frm**$(O(Y),O(X))$*
- *$A \cong$ **Frm**$(O(\mathbb{C}),O(X))$ classically equivalent to $A \cong C(X, \mathbb{C})$, **and** constructively valid provided we allow $O(X)$ to be an "arbitrary" (i.e. not necessarily spatial) frame*
- *Constructive Gelfand spectrum "X" of commutative C*-algebra A is **pointfree** space, i.e., object in $(\text{frames})^{op}$*
- *Typical situation in constructive mathematics!*

Intermezzo: topos theory

- *Topos theory is generalization of set theory*
- *Sets assemble into category* **Sets** *of sets and functions*
- *Topos is category in which one can do mathematics "as if it were set theory" except that all argument must be* ***constructive:***

  No axiom of choice, no law of excluded third

- *First examples of toposes due to Grothendieck (algebraic geometry)*
- *Axiomatization by Lawvere & Tierney: topos is category with terminal object, pullbacks, exponentials, and subobject classifier*
- *Foundations of classical mechanics (Lawvere, Bell)*
- *Foundations of quantum mechanics (Isham & Butterfield, Nijmegen group)*

Noncommutative Gelfand spectrum

- *A = unital C*-algebra (in* ***Sets****, or even in some other topos)*
- *Poset* ***C****(A) of unital commutative *-subalgebras of A*
- *Topos* $\mathbf{Sets}^{\mathbf{C}(A)}$ *of functors* $\underline{F}$*:* ***C****(A)* → ***Sets*** *[****C****(A) seen as category]* ≅

  *topos Sh(****C****(A)) of sheaves [****C****(A) seen as space in Alexandrov topology]*
- *"Tautological" functor* $\underline{A}$*: C*↦ *C (on arrows, C* ≤ *D* ↦ *i: C* ↪ *D)*

  This functor $\underline{A}$ *is a unital* ***commutative*** *C*-algebra in the topos T(A)*

  ➡ $\underline{A}$ *has (pointfree) Gelfand spectrum* $\underline{X}$ *in Sh(****C****(A)):* $\underline{X}$ *is itself a sheaf*
- ***Correspondence between noncommutative geometry and topos theory***

External description

- *Pointfree spaces $\underline{X}$ in sheaf toposes Sh(Y) have "external description" in set theory (Fourman-Scott, Joyal-Tierney, 1980):*

  *$\underline{X}$ in Sh(Y) ≅ frame map O(Y) → O(X), for some frame O(X) in **Sets***

  ➡ *Gelfand spectrum $\underline{X}$ of C*-algebra $\underline{A}$ in topos Sh(**C**(A)) ≅ frame map O(**C**(A)) → O(X) in **Sets**, for frame O(X) defined by $\underline{X}$*

  *"External Gelfand spectrum" O(X) of A computable if lattice of projections **P**(A) of A generates A (e.g., A is Rickart C*-algebra)*

  ➡ *O(X) = {S: **C**(A) → **P**(A) | S(C) ∈ **P**(C), S(C) ≦ S(D) if C ⊆ D}*

  *lattice w.r.t. **pointwise order** i.e. S ≤ T iff S(C) ≦ S(T) in **P**(A)*

- ***Lattice O(X) is (intuitionistic) logical description of C*-algebra A***

Two great Australians