Conley Index Theory and Condensed Sets

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Disclaimer / Confession

1) I will not use any deep results in condensed mathematics.

I even will not use profinite sets at all.

2) I will not use the language of ∞-categories.

It is simply because I am not fluent in it yet.

The ∞-categorical formulation should provide better results and should not be so difficult (?)

§ 1 Introduction

Conley index theory ⋯ a refinement / generalization of (some aspects of) Morse theory.

(Continuous time case : Conley '78, Salamon '85, ...
Discrete time case : Robbin - Salamon '88,
Mrozek '90, '94, Szymczak '95, Frank - Richeson '00, ...)

① $\boxed{\text{Morse index} \cdots \text{a natural number}}$ $\quad n$

$\rightsquigarrow$ refine $\quad \updownarrow$

$\boxed{\text{Conley index} \cdots \text{a space}}$ $\quad S^n$

(at least homotopically)

② Morse (- Bott) theory

... { · gradient flows on manifolds
· nondegenerate critical points (submanifolds)

more general

Conley index theory

... { · (partial) semiflows / self-maps on topological spaces
· isolated invariant subsets

Conley index theory is used not only in the study of topological dynamics themselves (pure and applied)

but also, for instance, in Manolescu's construction ('03) of the Seiberg-Witten-Floer stable homotopy type of 3-manifolds, a spatial refinement of the monopole Floer homology.

(far beyond the scope of today's talk)

There are some unsatisfactory features in the traditional formulation of Conley index theory :

- Proofs are complicated and does not provide conceptual understanding (at least to me).
- The Conley index in a single dynamical system is defined as a homotopy type of spaces (in the continuous time case ; even less data in the discrete time case).

Even worse, it is a mere homotopy type, i.e. homotopy coherence is not considered.

We want to define it as an actual space !

Two years ago,

I proposed a new framework for Conley index theory,

which I think is simpler and more flexible

than the traditional formulation.

My Conley index takes values in a rather mysterious category,

which I called $\mathrm{Sz}(\mathrm{CHaus}_*)$ (or $\mathrm{Sz}_{\mathrm{cont}}(\mathrm{CHaus}_*)$) in my paper.

named after [Szymczak '95]

I knew that it is the (1-categorically) universal receiver of

the construction, and that it has non-homotopical data.

However, I could not figure out what is it, after all.

Some months later, I realized that

$S_{\mathbb{Z}}(\mathrm{CHaus}_*)$ is naturally regarded as a full subcategory of

the category of based $\mathbb{Z}$- (or $\mathbb{R}$-) equivariant condensed sets.

So, my Conley index is an actual space, as I wanted!

In today's talk, I will explain

my latest understanding of the Conley index.

(⚠ { • Still not in a definitive form.
 • The traditional formulation not explained.)

For simplicity, I will concentate on the discrete time case.

§ 2 Partial maps on locally compact Hausdorff spaces

Def^n : X, Y : LCHaus spaces.

A (continuous) partial map from X to Y is a diagram

$$X \hookleftarrow \mathrm{Dom}\, f \xrightarrow{f} Y, \quad \text{where}$$

$$\begin{cases} \bullet\ \mathrm{Dom}\, f \hookrightarrow X : \text{a locally closed embedding} \\ \qquad\qquad \Updownarrow \\ \qquad (\mathrm{Dom}\, f : \text{LCHaus}) \\ \bullet\ \mathrm{Dom}\, f \xrightarrow{f} Y : \text{a continuous map.} \end{cases}$$

We will regard $\mathrm{Dom}\, f$ as a subset of X

Partial maps will be denoted as $f : X \rightharpoonup Y$.

For $f : X \to Y$ and $g : Y \to Z$,

we can define the composite $g \circ f : X \longrightarrow Z$.

$$\left(\mathrm{Dom}(g \circ f) := f^{-1}(\mathrm{Dom}\, g) \right).$$

Def^n : (1) $f : X \longrightarrow Y$ is proper

$\overset{\text{def.}}{\Longleftrightarrow}$ $f : \mathrm{Dom} f \longrightarrow Y$ is a proper map.

(2) f : openly defined

$\overset{\text{def.}}{\Longleftrightarrow}$ $\mathrm{Dom} f \hookrightarrow X$ is an open embedding.

They are closed under composition.

If $f : X \to Y$ is proper and openly defined,

the map
$$f^+ : X^+ \longrightarrow Y^+$$
$$x \longmapsto \begin{cases} f(x) & (x \in \mathrm{Dom}\, f) \\ * & (\text{otherwise}) \end{cases}$$
is continuous.

(one-point compactification. $Y^+ = Y \sqcup \{*\}$ as a set)

Conversely, if $g : (K, x_0) \to (L, y_0)$ is a based continuous map of based CHaus spaces,

then $g^- : K \setminus \{x_0\} \to L \setminus \{y_0\}$ defined by

$\mathrm{Dom}\, g^- := g^{-1}(L \setminus \{y_0\})$, $g^- := g$ on $\mathrm{Dom}\, g^-$

is proper and openly defined.

$\underline{\mathrm{LCHaus}^{\partial}}$: the category of locally compact Hausdorff spaces and proper openly defined partial maps.

$\underline{\mathrm{CHaus}_*}$: the category of based CHaus spaces and based continuous maps.

The one-point compactification induces an equivalence $\mathrm{LCHaus}^{\partial} \simeq \mathrm{CHaus}_*$.

In the rest of my talk, we fix

a continuous partial self-map $f : X \longrightarrow X$.

(… a discrete time topological (semi)dynamical system)

<u>Def^n</u> : For $E \subset X$: locally closed,

define <u>f_E</u> $: E \longrightarrow E$ by

$\mathrm{Dom}\, f_E := E \cap f^{-1}(E)$, $f_E := f$ on $\mathrm{Dom}\, f_E$.

(Even if $f : X \longrightarrow X$ is a homeomorphism,
$f_E : E \to E$ is usually <u>not</u> everywhere defined.)

Def^{n} : For $E, E' \subset X$: locally closed.

$E \sim_f E'$ $\overset{\text{def.}}{\iff}$ For $a, b, a', b' \gg 0$,

$$\begin{cases} \cdot \ \bigcap_{i=0}^{a+b} f^{-i}(E) \subset f^{-a}(E'). \\ \cdot \ \bigcap_{i'=0}^{a'+b'} f^{-i'}(E') \subset f^{-a'}(E). \end{cases}$$

(If your point stays in E from time 0 to time $a+b$, then it must be in E' at time a.)

Thus, informally speaking, $E \sim_f E'$ means that

'E and E' has the same size modulo the dynamics f'.

Def^n : For $E, E' \subset X$: locally closed and $a, b, c \in \mathbb{N}$,

define $f_{E'E}^{(a,b,c)} : E \longrightarrow E'$ by

$$\begin{cases} \cdot \ \operatorname{Dom} f_{E'E}^{(a,b,c)} := \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E'), \\ \cdot \ f_{E'E}^{(a,b,c)} := f^{a+b+c} \quad \text{on} \quad \operatorname{Dom} f_{E'E}^{(a,b,c)}. \end{cases}$$

By definition, $f_{EE}^{(a,b,c)} = f_E^{a+b+c}$.

(**Notation** : f_E^n always means $(f_E)^n$, not $(f^n)_E$.
Thus, $\operatorname{Dom} f_E^n = \bigcap_{i=0}^{n} f^{-i}(E)$.)

The first key result is the following :

$$\left[\begin{array}{l}\underline{\text{Th}^{\text{m}}\text{ A}} : \quad E \sim_f E' \sim_f E'' . \qquad a.b.c.\ a'.b'.c' \gg 0 \\ \Longrightarrow \quad f_{E''E'}^{(a'.b'.c')} \circ f_{E'E}^{(a.b.c)} = f_{E''E}^{(a+a'.\ b+b'.\ c+c')}\end{array}\right]$$

(The nontrivial point is that they have the same domain !)

The proof is elementary and completely formal

(computation of subset inclusions ; omitted).

As a corollary, since $f_{EE}^{(a.b.c)} = f_E^{a+b+c}$, we have :

$$\begin{cases} \cdot \ f_{E'E}^{(a.b.c)} \circ f_E = f_{E'} \circ f_{E'E}^{(a.b.c)} & (a.b.c \gg 0), \\ \cdot \ f_{E'E}^{(a.b.c)} \circ f_E^{a'+b'+c'} = f_{E'E}^{(a'.b'.c')} \cdot f_E^{a+b+c} & (a.b.c.\ a'.b'.c' \gg 0). \end{cases}$$

To state the second key result,

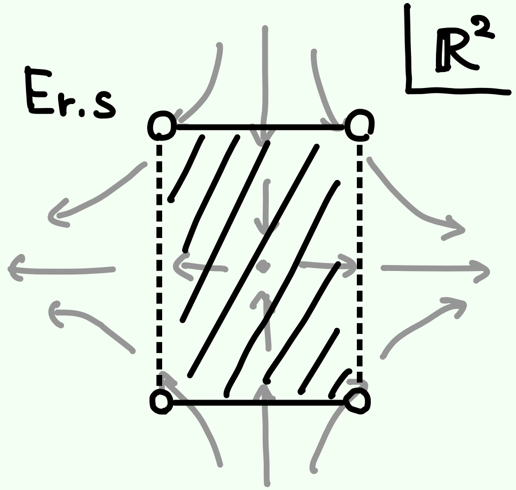
we introduce the following terminology:

Def^n: A locally closed subset $E \subset X$ is <u>f-compactifiable</u>

$\overset{\text{def.}}{\Longleftrightarrow}$ $f_E : E \longrightarrow E$ is proper and openly defined.

($\rightsquigarrow$ $f_E^+ : E^+ \longrightarrow E^+$ a based continuous self-map on a based CHaus space E^+)

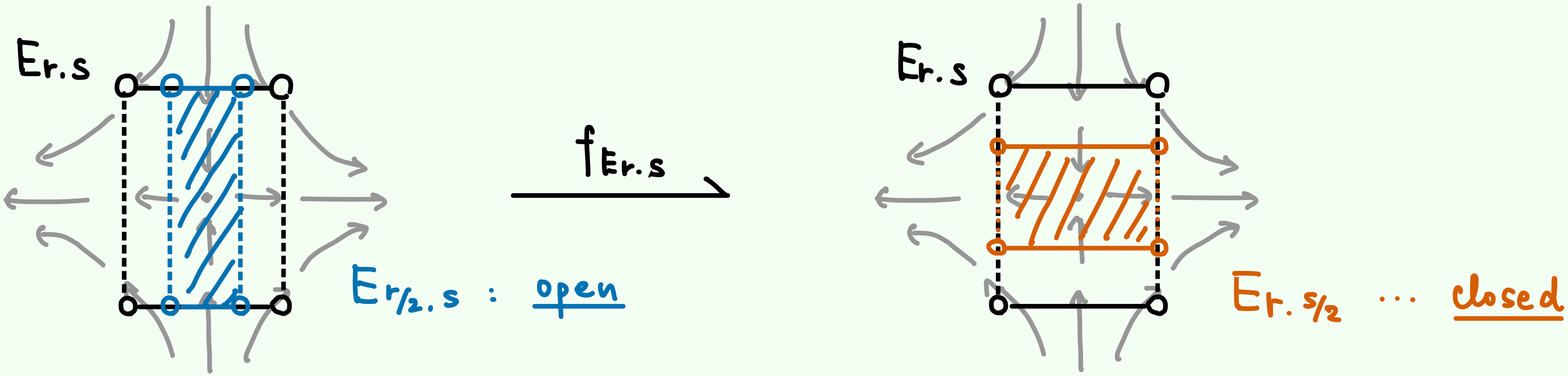
Ex : $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\quad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2x \\ y/2 \end{pmatrix}$.



$$E_{r,s} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; |x| < r,\ |y| \leq s \right\} \quad (r,s > 0)$$

is an f-compactifiable subset of $\mathbb{R}^2$:

- $\operatorname{Dom} f_{E_{r,s}} = E_{r/2,s} \underset{\text{open}}{\subset} E_{r,s}$.
- $f_{E_{r,s}} : E_{r/2,s} \longrightarrow E_{r,s}$ is a closed embedding with image $E_{r,s/2} \underset{\text{closed}}{\subset} E_{r,s}$, hence proper.



$\underline{\text{Th}^{\text{m}}\ \text{B}}$: $E \sim_f E'$. $a.b.c \gg 0$.

(1) $f_E : E \longrightarrow E$, $f_{E'} : E' \longrightarrow E'$ proper

$\Longrightarrow f^{(a.b.c)}_{E'E} : E \longrightarrow E'$ proper.

(2) $f_E : E \longrightarrow E$, $f_{E'} : E' \longrightarrow E'$ openly defined

$\Longrightarrow f^{(a.b.c)}_{E'E} : E \longrightarrow E'$ openly defined

(In particular, if E and E' are f-compactifiable, $f^{(a.b.c)}_{E'E} : E \to E'$ is proper and openly defined.)

The proof is again elementary and completely formal.

We give the proof of (1) only. (2) is similarly proved.

Proof of Th^{m} B (1) : We can factorize

$$f_{E'E}^{(a.b.c)} : \mathrm{Dom}\, f_{E'E}^{(a.b.c)} \longrightarrow E' \quad \text{as follows :}$$

$$\mathrm{Dom}\, f_{E'E}^{(a.b.c)} = \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E')$$

$$\xrightarrow[\text{①}]{f_E^a} \bigcap_{i=0}^{b} f^{-i}(E) \cap \bigcap_{i'=0}^{b+c} f^{-i'}(E')$$

$$\overset{\text{②}}{\hookrightarrow} \bigcap_{i'=0}^{b+c} f^{-i'}(E') \xrightarrow[\text{③}]{f_{E'}^{b+c}} E'.$$

It suffices to show that ①, ②, ③ are all proper.

⋮

⋮

3 : $f'_E : E' \longrightarrow E'$ is proper

$\rightsquigarrow$ 3 $= \underbrace{f_{E'} \circ \cdots \circ f_{E'}}_{b+c} : E' \longrightarrow E'$ is proper. ⌟

1 : The following is a pullback diagram :

$$\begin{tikzcd}
\bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E') \arrow[r, "f_E^a", "1"'] \arrow[d, hook] & \bigcap_{i=0}^{b} f^{-i}(E) \cap \bigcap_{i'=0}^{b+c} f^{-i'}(E') \arrow[d, hook] \\
\bigcap_{i=0}^{a} f^{-i}(E) \arrow[r, "f_E^a", "1'"'] & E
\end{tikzcd}$$

$f_E : E \longrightarrow E$ proper $\rightsquigarrow$ 1' is proper

$\rightsquigarrow$ 1 is proper. ⌟

⋮

⋮

2 : The following is a pullback diagram :

$$\begin{tikzcd}
\bigcap_{i=0}^{b} f^{-i}(E) \cap \bigcap_{i'=0}^{b+c} f^{-i'}(E') \arrow[r, hook, "2"'] \arrow[d, hook] & \bigcap_{i'=0}^{b+c} f^{-i'}(E') \arrow[d, hook] \\
\bigcap_{i=0}^{b} f^{-i}(E) \arrow[r, hook, "2'"'] & f^{-b}(E)
\end{tikzcd}$$

(This inclusion holds since $E \sim_f E'$ and $b, c \gg 0$!)

To see that 2 is proper,

it is enough to show that 2' is proper.

⋮

$$
\begin{tikzcd}
\bigcap_{i=0}^{b} f^{-i}(E) \arrow[r, hook, "2'"] \arrow[dr, "f_E^b"', "2''"] & f^{-b}(E) \arrow[d, "f^b"] \arrow[dl, phantom, "\circlearrowleft"] \\
& E
\end{tikzcd}
$$

$f_E : E \longrightarrow E$ is proper $\rightsquigarrow$ $2''$ is proper

$\rightsquigarrow$ $2'$ is proper $\lrcorner$

(In LCHaus, every morphism is separated)

This completes the proof. $\blacksquare$

§3 Based $\mathbb{Z}$-equivariant condensed sets associated with f-compactifiable subsets

We now want to explain an implication of Th^{ms} A and B.

But before doing so,

let us introduce the following general construction :

Def^n : $\mathcal{C}$: a category

$u : A \longrightarrow A$ an endomorphism in $\mathcal{C}$.

Define $u_\infty : A_\infty \longrightarrow A_\infty$ in $\mathrm{Ind}(\mathcal{C})$ by

(ind - category)

- $A_\infty := \text{` } \varinjlim \text{ '} \left(A \xrightarrow{u} A \xrightarrow{u} A \xrightarrow{u} \cdots \right)$

(' ' signifies the colimit in $\mathrm{Ind}(\mathcal{C})$)

- u_∞ is induced from u :

$$
\begin{array}{c} A_\infty \\ \big\downarrow u_\infty \\ A_\infty \end{array}
:=
\left[
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "u"] & A \arrow[r, "u"] \arrow[d, "u"] & A \arrow[r, "u"] \arrow[d, "u"] & \cdots \\
A \arrow[r, "u"] & A \arrow[r, "u"] & A \arrow[r, "u"] & \cdots
\end{tikzcd}
\right]
$$

[**Lem**: $u_\infty : A_\infty \xrightarrow{\cong} A_\infty$ is an isomorphism.]

Pf: The inverse is explicitly given by

$$
\begin{tikzcd}
A_\infty \\
A_\infty \arrow[u, "u_\infty^{-1}"]
\end{tikzcd}
=
\left[
\begin{tikzcd}
A \arrow[r, "u"] & A \arrow[r, "u"] & A \arrow[r, "u"] & \cdots \\
A \arrow[r, "u"] \arrow[ur, "\mathrm{id}_A"] & A \arrow[r, "u"] \arrow[ur, "\mathrm{id}_A"] & A \arrow[r, "u"] \arrow[ur, "\mathrm{id}_A"] & \cdots
\end{tikzcd}
\right]
$$

$\blacksquare$

(**Recall**:
$$
\begin{tikzcd}
A_\infty \arrow[d, "u_\infty"] \\
A_\infty
\end{tikzcd}
:=
\left[
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "u"] & A \arrow[r, "u"] \arrow[d, "u"] & A \arrow[r, "u"] \arrow[d, "u"] & \cdots \\
A \arrow[r, "u"] & A \arrow[r, "u"] & A \arrow[r, "u"] & \cdots
\end{tikzcd}
\right]
$$
)

$\rightsquigarrow$ A_∞ is a <u>$\mathbb{Z}$-equivariant</u> ind-object of $\mathcal{C}$.

An interpretation :

$u : A \longrightarrow A$ ... a (semi) dynamical system.

u is (usually) not a monomorphism.

$\rightsquigarrow$ We lose some information as time passes.

$$A_\infty := \text{`}\varinjlim\text{'} \left(A \xrightarrow{u} A \xrightarrow{u} A \xrightarrow{u} \cdots \right)$$

knows exactly the 'long time behaviour',

i.e. all information which is not lost within finite time.

Now, let us go back to the setting of Th$^{\text{ms}}$ A and B.

$E \subset X$: f-compactifiable

$\rightsquigarrow$ $f_E : E \longrightarrow E$ proper, openly defined.

Equivalently, $f_E^+ : E^+ \longrightarrow E^+$ in CHaus_*

$\rightsquigarrow$ $E_\infty^+ := \text{“}\varinjlim\text{”} \left(E^+ \xrightarrow{f_E^+} E^+ \xrightarrow{f_E^+} E^+ \xrightarrow{f_E^+} \cdots \right)$

$(f_E^+)_\infty : E_\infty^+ \xrightarrow{\cong} E_\infty^+$

... an isomorphism in $\mathrm{Ind}(\mathrm{CHaus}_*) \simeq \mathrm{Ind}(\mathrm{CHaus})_*$.

Using Th^{ms} A and B, we can define

$\varphi_{E'E} : E_\infty^{+} \longrightarrow E'^{+}_\infty$ for $E \sim_f E'$ as follows:

① Fix $a, b, c \gg 0$.

② Since $f_{E'E}^{(a,b,c)} \circ f_E = f_{E'} \circ f_{E'E}^{(a,b,c)}$

and since $f_{E'E}^{(a,b,c)}$ is proper and openly defined,

$\left(f_{E'E}^{(a,b,c)}\right)_\infty^{+} : E_\infty^{+} \longrightarrow E'^{+}_\infty$ is induced.

③ Put $\varphi_{E'E} := \left(f_{E'E}^{(a,b,c)}\right)_\infty^{+} \circ \left(f_E^{+}\right)_\infty^{-(a+b+c)}$.

($\varphi_{E'E}$ does not depend on the choice of $a, b, c \gg 0$

since $f_{E'E}^{(a,b,c)} \circ f_E^{a'+b'+c'} = f_{E'E}^{(a',b',c')} \circ f_E^{a+b+c}$.)

Properties of $\varphi_{E'E}$:

- Obviously, $\varphi_{EE} = \mathrm{id}_{E^+_\infty}$.
- Since $f^{(a',b',c')}_{E''E'} \circ f^{(a,b,c)}_{E'E} = f^{(a+a',b+b',c+c')}_{E''E}$,

  we have $\varphi_{E''E'} \circ \varphi_{E'E} = \varphi_{E''E}$.

  ($\rightsquigarrow$ $\varphi_{E'E}$ is an isomorphism whose inverse is $\varphi_{EE'}$)

- Since $f^{(a,b,c)}_{E'E} \circ f_E = f_{E'} \circ f^{(a,b,c)}_{E'E}$,

  we have $\varphi_{E'E} \circ (f^+_E)_\infty = (f^+_{E'})_\infty \circ \varphi_{E'E}$.

  (i.e. $\varphi_{E'E}$ is $\mathbb{Z}$-equivariant).

Summary :

- Each $\dagger$-compactifiable subset $E \subset X$ defines an object E_∞^+ of $\underset{\sim}{\mathbb{Z}}$ - Ind(CHaus)$_*$.
  ($\mathbb{Z}$-equivariant).

- If $E \sim_f E'$, we obtain an isomorphism $\varphi_{E'E} : E_\infty \xrightarrow{\cong} E'_\infty$ in $\mathbb{Z}$ - Ind(CHaus)$_*$.

- The identification of E_∞ and E'_∞ via $\varphi_{E'E}$ is legitimate, i.e.

$$
\begin{tikzcd}
E_\infty \arrow[rr, "\varphi_{E''E}", "\cong"'] \arrow[dr, "\cong", "\varphi_{E'E}"'] & \circlearrowleft & E''_\infty \\
& E'_\infty \arrow[ur, "\cong"', "\varphi_{E''E'}"] &
\end{tikzcd}
$$

.

Now, recall that Ind(CHaus) is a full subcategory of

the category <u>Cond(Set)</u> of <u>condensed sets</u> !

Hence, $\mathbb{Z}$-Ind(CHaus)$_*$ is a full subcategory of

the category <u>$\mathbb{Z}$-Cond(Set)$_*$</u> of

<u>based $\mathbb{Z}$-equivariant condensed sets</u> .

($\rightsquigarrow$ Each $\sim_f$-equivalence class of f-compactifiable subsets

defines a based $\mathbb{Z}$-equivariant condensed set

up to unique isomorphism !)

§4 The Conley index of isolated f-invariant subsets.

How to find f-compactifiable subsets ?

Th^{m} C : $E, E' \subset X$: locally closed, $E \sim_f E'$.

$f_E : E \longrightarrow E$ proper, $f_{E'} : E' \longrightarrow E'$ openly defined

$\Longrightarrow$ For $a, b, c \gg 0$.

$$E'' := \bigcap_{i=0}^{a+b} f^{-i}(E) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(E')$$

is f-compactifiable and $E'' \sim_f E \ (\sim_f E')$.

Once again, the proof is elementary and completely formal

(omitted)

As an application of Thm C, let us define the Conley index of isolated f-invariant subsets.

Def^n: $S \subset X$ is f-invariant

$\overset{\text{def.}}{\Longleftrightarrow}$ $S \subset \mathrm{Dom}\, f$ and $f(S) = S$.

(⚠ stronger than $S \subset f^{-1}(S)$.)

Def^n: For $E \subset X$, define its f-invariant part $I_f(E)$ to be the largest f-invariant subset of E.

Explicitly, $I_f(E) = \bigcap_{a,b \in \mathbb{N}} f^a\left(\bigcap_{i=0}^{a+b} f^{-i}(E) \right)$ (easy)

Def^n : $S \subset X$: f-invariant.

(1) A locally closed neighbourhood E of S is isolating

$\overset{\text{def.}}{\Longleftrightarrow}$
- $\overline{E}$ is compact and $\overline{E} \subset \mathrm{Dom}\, f$.
- $S = I_f(\overline{E})$. (the closure of E)

(2) S is isolated

$\Longleftrightarrow$ $\exists$ an isolating neighbourhood of S.

Prop. D : $S \subset X$: isolated f-invariant.

E, E' : isolating neighbourhoods of S $\Longrightarrow$ $E \sim_f E'$.

Pf : Put $K := \overline{E}$, $U' := \mathring{E'}$. ← the interior of E'

Then, $\emptyset = (K \setminus U') \cap S$

$$= \underbrace{(K \setminus U')}_{\text{closed in } K} \cap \overbrace{\bigcap_{a,b \in \mathbb{N}} \underbrace{f^{a}\left(\bigcap_{i=0}^{a+b} f^{-i}(K) \right)}_{\text{closed in } K}}^{\text{decreasing family}} .$$

Since K is compact. $\exists\, a_0, b_0 \in \mathbb{N}$ s.t.

$$(K \setminus U') \cap f^{a_0}\left(\bigcap_{i=0}^{a_0+b_0} f^{-i}(K) \right) = \emptyset .$$

i.e.
$$\begin{array}{ccc} \underline{\bigcap_{i=0}^{a_0+b_0} f^{-i}(K)} & \subset & \underline{f^{-a_0}(U')} \\ \cup & & \cap \\ \bigcap_{i=0}^{a_0+b_0} f^{-i}(E) & & f^{-a_0}(E'). \end{array}$$

The same for the converse direction. $\square$

Def^n : $S \subset X$: isolated f-invariant

A locally closed neighbourhood E of S is an index neighbourhood

$\overset{\text{def.}}{\iff}$ $\begin{cases} \cdot \ E \text{ is an isolating neighbourhood of } S. \\ \cdot \ E \text{ is } f\text{-compatifiable.} \end{cases}$

Prop. E : $S \subset X$: isolated f-invariant

$\Rightarrow$ The set of all index neighbourhoods forms a neighbourhood basis of S.

(In particular, $\exists$ an index neighbourhood of S)

Pf : S : isolated $\Longrightarrow$ S : compact (easy)

Thus, it is enough to show that

$\forall K$: compact isolating neighbourhood of S,

$\exists E$: an f-compactifiable neighbourhood of S

s.t. $E \subset K$.

Take any open isolating neighbourhood U of S,

(e.g. $U := \mathring{K}$)

$\rightsquigarrow$ By Prop. D, $K \sim_f U$.

Take $a, b, c \gg 0$ and put

$$E := \bigcap_{i=0}^{a+b} f^{-i}(K) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(U).$$

⋮

⋮

$$E := \bigcap_{i=0}^{a+b} f^{-i}(K) \cap \bigcap_{i'=a}^{a+b+c} f^{-i'}(U) \qquad (a.b.c \gg 0).$$

E is a neighbourhood of S, and $E \subset K$.

- $f_K : K \longrightarrow K$ is proper.

  ($\mathrm{Dom}\, f_K = K \cap f^{-1}(K)$ is a CHaus space)

- $f_U : U \longrightarrow U$ is openly defined.

  ($\mathrm{Dom}\, f_U = U \cap f^{-1}(U)$ is open in U)

$\leadsto$ By Th^m C, E is f-compactifiable. $\square$

Def^n : $S \subset X$: isolated f-invariant

We define the Conley index $\mathrm{Ind}_f(S)$ of S

to be the based $\mathbb{Z}$-equivariant condensed set

$$\mathrm{Ind}_f(S) := E^+_\infty = \text{`}\varinjlim\text{'}\left(E^+ \xrightarrow{f_E^+} E^+ \xrightarrow{f_E^+} \cdots \right)$$

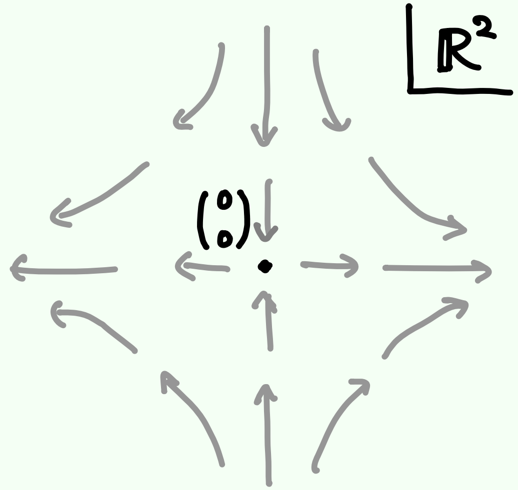
where E is an index neighbourhood of S.

(always exists by Prop. E)

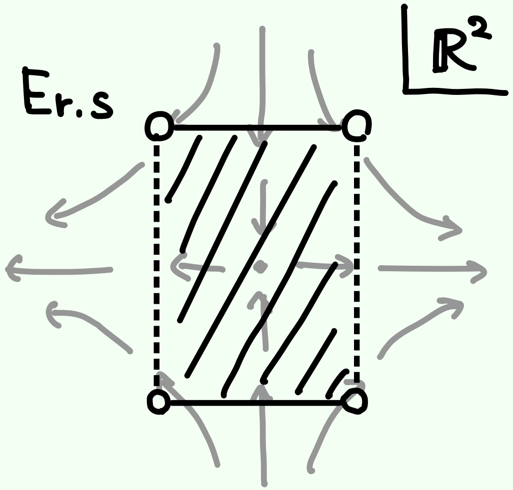
By Prop. D, $\mathrm{Ind}_f(S)$ does not depend on

the choice of E up to unique isomorphism!

__Ex__ : $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2x \\ y/2 \end{pmatrix}.$



$S := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$: isolated f-invariant



$$E_{r,s} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; |x| < r,\ |y| \leq s \right\} \quad (r,s > 0)$$

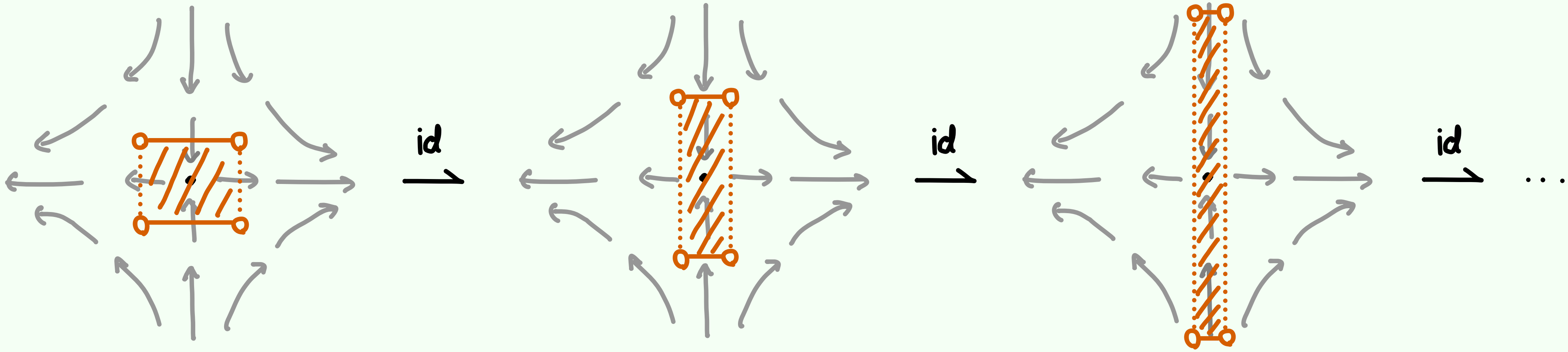
is an index neighbourhood of S.

$$\mathrm{Ind}_f(S) := \text{`}\varinjlim\text{'} \left(E_{r,s}^+ \xrightarrow{f^+_{E_{r,s}}} E_{r,s}^+ \xrightarrow{f^+_{E_{r,s}}} E_{r,s}^+ \xrightarrow{f^+_{E_{r,s}}} \cdots \right)$$

(__Rmk__ : $E^+_{r,s}$ is homotopy equivalent to S^1 (Morse index 1) and $f^+_{E_{r,s}}$ is a homotopy equivalence.)

$$\mathrm{Ind}_f(S) = \text{`}\varinjlim\text{'}\left(E_{r,s}^{+} \xrightarrow{f_{E_{r,s}}^{+}} E_{r,s}^{+} \xrightarrow{f_{E_{r,s}}^{+}} E_{r,s}^{+} \xrightarrow{f_{E_{r,s}}^{+}} \cdots \right)$$

$$\cong \text{`}\varinjlim\text{'}\left(E_{r,s}^{+} \xrightarrow{\mathrm{id}^{+}} E_{r/2,2s}^{+} \xrightarrow{\mathrm{id}^{+}} E_{r/4,4s}^{+} \xrightarrow{\mathrm{id}^{+}} \cdots \right)$$



In this case, the Conley index looks quite similar to the Thom space for the normal bundle of the stable submanifold!

(The same for the other Morse-like cases)

Future directions / Speculations :

- Homotopy invariance of the Conley index.
- Rewriting the theory of connection matrices.

  (Conley index version of Morse homology / homotopy)
- Application to Floer theory ?
- Categories other than LCHaus ?

  (algebro - geometric / arithmetic setting ?)
- Possible relation to the six - functor formalism ?

  ($\leftrightarrow$ the case where E is not f - compactifiable ?)
- Hyperbolic localization ?

Thank you !!