Lecture Notes on Dualizable Categories and Localizing Invariants.
Akhil Mathew, MPIM Bonn.

Texed by Chirantan Chowdhury

September 2024

Contents

1 Introduction

These are live latex notes of the lecture series given by Akhil Mathew as a part of the workshop "Dualizable Categories and Continuous K-theory" held at MPIM Bonn (Sept 9-13, 2024).

The lecture series cover an introduction to the notion of Dualizable categories and localizing invariants which is useful in defining K-theory for large presentable categories extending the notion from stable ∞-categories. The lecture series include defining the basic notions in these contexts and stating results on how to compute limits and colimits in the dualizable context.

Apologies in advance for typos and mistakes. I hope these lecture notes are helpful for reader who are looking for a concise introduction to the language of dualizable categories.

2 Lecture 1 : Localizing Invariants.

Setup: Let $\mathcal{C}$ be a small stable ∞-category.

Definition 2.1. $\mathsf{K}_0(\mathcal{C})$ is defined as free abelian group on $[\mathsf{X}] \in \mathcal{C}$ modulo the relation of $[\mathsf{X}] = [\mathsf{X}'] + [\mathsf{X}'']$ where $\mathsf{X}' \to \mathsf{X} \to \mathsf{X}''$ is a fiber sequence.

The above notion depends on the homotopy category.

We can define after Quillen, Waldhausen a spectrum $\mathsf{K}(\mathcal{C})$ such that $\pi_0\mathsf{K}(\mathcal{C})$ is $\mathsf{K}_0(\mathcal{C})$. $\mathsf{K}(\mathcal{C})$ is homotopical enhancement of K_0.

Definition 2.2. $\mathrm{Cat}^{\mathrm{perf}}$ is the ∞-category of small, idempotent complete stable ∞-categories and exact functors between theme which preserve finite colimits.

Definition 2.3. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ and $\mathcal{D} \subset \mathcal{C}$ inclusion in $\mathrm{Cat}^{\mathrm{perf}}$. Then the *Karoubi* quotient is the pushout

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{C}/\mathcal{D} \end{array} \tag{1}$$

which is a pushout and it is infact a pullback. The sequence $0 \to \mathcal{D} \to \mathcal{C} \to \mathcal{C}/\mathcal{D} \to 0$ is called a *Karoubi sequence.*

Informal description of the Karoubi quotient: We have a quotient functors

$$\mathsf{p} : \mathcal{C} \to \mathcal{C}/\mathcal{D}$$

Given $\mathsf{X}, \mathsf{Y} \in \mathcal{C}$, then

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{D}}(\mathsf{X}, \mathsf{Y}) = \mathrm{colim}_{\mathsf{Y}'} \mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y}')$$

where the colimits is over all such Y' such. that $\mathrm{cofib}(\mathsf{Y} \to \mathsf{Y}') \in \mathcal{D}$.

Remark 2.4. The above morphism p is not essentially surjective but it is upto retracts.

Example 2.5.

- Let X be a qcqs scheme and $\mathsf{U} \subset \mathsf{X}$ qc open. Then the we have an exact sequence :

  $$\mathrm{Perf}(\mathsf{X} \text{ on } \mathsf{Z}) \to \mathrm{Perf}(\mathsf{X}) \to \mathrm{Perf}(\mathsf{U})$$

  is a Karoubi sequence due to Thomasson-Trobaugh.

- Let $\mathcal{A} \in \mathrm{Cat}^{\mathrm{perf}}$. Then define

  $$\mathrm{Calk}(\mathcal{A}) = \mathrm{Ind}(\mathcal{A})/\mathcal{A} \quad ; \quad \mathrm{Calk}_{\kappa}(\mathcal{A}) = \mathrm{Ind}^{\kappa}(\mathcal{A})/\mathcal{A} \tag{2}$$

  where κ is a regular cardinal.
  Note that for representative

  $$\mathrm{Hom}_{\mathrm{Calk}(\mathcal{A})}(\text{"}\varinjlim_{i} \mathsf{X}_i\text{"}, \text{"}\varinjlim_{j} \mathsf{Y}_j\text{"}) = \frac{\varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{A}}(\mathsf{X}_i, \mathsf{Y}_j)}{\varinjlim_j \varprojlim_i \mathrm{Hom}_{\mathcal{A}}(\mathsf{X}_i, \mathsf{Y}_j)}$$

- If $\mathcal{A} = \mathrm{Perf}(\mathsf{R})$, then $\mathrm{Calk}(\mathcal{A}) = \mathsf{D}(\mathsf{R})/\mathrm{Perf}(\mathsf{R})$.

- If k is a field and V is a vector space and let $\mathcal{A}$ = category of vector spaces over k, then $\mathrm{Calk}(\mathcal{A})$ is the quotient

  $$\frac{\mathsf{k}-\text{linear maps}}{\text{maps of finite rank}}. \tag{3}$$

We recall the notion of semi-orthogonal. decompsition which is an alternate way of understanding Karoubi sequences.

Definition 2.6. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{stb,idem}}$. A *semiorthogonal decomposition of* $\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ consists of stable subcats $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{C}$ such that

1. $X_1 \in \mathcal{C}_1, X_2 \in \mathcal{C}_2 \implies \mathrm{Hom}_{\mathcal{C}}(X_2, X_1) = 0$,
2. If $Y \in \mathcal{C}$, then there exists a fiber sequence
$$Y_2 \to Y \to Y_1 \tag{4}$$
such that $Y_1 \in \mathcal{C}_1, Y_2 \in \mathcal{C}_2$.

Remark 2.7. Some remarks on the above definition:

- Y determines Y_1, Y_2 upto contractible choices.
- Enough to require that $\mathcal{C}_1, \mathcal{C}_2$ generate $\mathcal{C}$.
- We have a Karoubi sequence
$$\mathcal{C}_2 \to \mathcal{C} \to \mathcal{C}_1 \tag{5}$$
where the latter map sends $Y \to Y_1$.
- An example is $\mathrm{Perf}(\mathbf{P}^1) = \langle \mathcal{O}, \mathcal{O}(1) \rangle$.
- Given a Karoubi sequence
$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{p} \mathcal{B}/\mathcal{A} \to 0 \tag{6}$$
if either
  1. i admits a right adjoint i^R
  2. p admits a right adjoint p^R.

  then $\mathcal{B} = \langle \ker(i^R), p(\mathcal{A}) \rangle$ or $\langle p^R(\mathcal{B}/\mathcal{A}), i(\mathcal{A}) \rangle$

Construction 2.8. Let $\mathcal{D} \subset \mathcal{C}$ in $\mathrm{Cat}^{\mathrm{perf}}$. We want
$$\mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C}/\mathcal{D})$$
to be Karoubi sequence. This forces
$$\mathcal{C}/\mathcal{D} = \ker(i^R : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D}))^{\omega}.$$

Remark 2.9. $\mathrm{Ind}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ with this identification i^R is am map given by restriction.

Definition 2.10. Let $\mathcal{E}$ be a stable ∞-category. A *localizing invariant* is a functor
$$F : \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E} \tag{7}$$
such that

1. $F(0) = 0$,
2. If $\mathcal{D} \subset \mathcal{C}$ an inclusion in $\mathrm{Cat}^{\mathrm{perf}}$, then
$$\begin{array}{ccc} F(\mathcal{D}) & \longrightarrow & F(\mathcal{C}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & F(\mathcal{C}/\mathcal{D}) \end{array} \tag{8}$$
is a pushout.

Usually we require $\mathcal{E}$ to be accessbile and F preserves κ-filtered colimits for some κ.

Example 2.11.
- If $\mathcal{C} = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ and F is localizing , then $\mathsf{F}(\mathcal{C}) \cong \mathsf{F}(\mathcal{C}_1) \oplus \mathsf{F}(\mathcal{C}_2)$. Functors satisfying this condition are called *additive invariants*.
- $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$, consider $\mathrm{Ind}^{\omega_1}(\mathcal{C})$ and $\mathrm{Calk}_{\omega_1}(\mathcal{C})$, then we have a fiber sequence

$$\mathsf{F}(\mathcal{C}) \to \mathsf{F}(\mathrm{Ind}^{\omega_1}(\mathcal{C})) \to \mathsf{F}(\mathrm{Calk}_{\omega_1}(\mathcal{C})) \tag{9}$$

  the middle term vanishes due to Eilenberg-Swindle and thus we have $\mathsf{F}(\mathcal{C}) \cong \Omega \mathsf{F}(\mathrm{Calk}_{\omega_1}(\mathcal{C}))$.

Theorem 2.12 (BGT,Barwick)**.** $\mathsf{K} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Sp}$ *is the initial localizing invariant equipped with a natural map* $\mathcal{C}^{\sim} \to \Omega^{\infty}\mathsf{K}(\mathcal{C})$.

Other examples of localizing invariants are $\mathrm{THH}, \mathrm{TC}$.

Definition 2.13 (BGT)**.** The ∞-category of *localizing motives* $\mathrm{Mot}_{\mathrm{loc}}$ is the initial presentable stable ∞-category equipped with a localizing invariant

$$\mathcal{U}_{\mathrm{loc}} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Mot}_{\mathrm{loc}} \tag{10}$$

that preserves filtered colimits.

Remark 2.14. $\mathrm{Mot}_{\mathrm{loc}}$ is quotient of $\mathrm{Fun}(\mathrm{Cat}^{\mathrm{perf}}, \mathrm{Sp})$.

Theorem 2.15 (BGT)**.** *For all* $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ *, we have*

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{loc}}}(\mathcal{U}_{\mathrm{loc}}(\mathrm{Sp}^{\omega}), \mathcal{U}_{\mathrm{loc}}(\mathcal{C})) \cong \mathsf{K}(\mathcal{C})$$

.

3 Lecture II : Dualizable categories.

Definition 3.1. An ∞-category $\mathcal{C}$ is *presentable* if

1. $\mathcal{C}$ has all colimits,
2. For some regular cardinal κ, $\mathcal{C}^{\kappa}$ is small and $\mathcal{C} \cong \mathrm{Ind}_{\kappa}(\mathcal{C}^{\kappa})$ where $\mathcal{C}^{\kappa}$ is all such objects $X \in \mathcal{C}$ such that $\mathrm{Hom}(-, X)$ preserves κ-filtered colimits.

Example 3.2. $\mathcal{C} = \mathrm{Ind}(\mathcal{C}^{\omega})$ is presentable. We will work on $\kappa = \omega_1$ which applies to most cases in practice this week. Commuting with ω_1-filtered colimits means that any countable subset has an upper bound (whereas filtered means only finite sets has upper bound).

Definition 3.3. Pr^{L} is ∞-category whose objects are presentable ∞-categories and morphisms are colimit preserving functors.

Remark 3.4.
- Limits in Pr^{L} are computed in big infinity categories via the inclusion $\mathrm{Pr}^{\mathsf{L}} \hookrightarrow \widehat{\mathrm{Cat}_{\infty}}$.
- Colimits in Pr^{L} are computed by taking limits in Pr^{R} via the inclusion to $\widehat{\mathrm{Cat}_{\infty}}$.
- An example of a colimit computation is that if $G \cong \mathrm{colim}\, G_i$, then G-sets is limit of G_i sets.
- Another example of this is in the previous lecture while forming the pushout $\mathcal{C}/\mathcal{D}$ (Eq. (1)). One can see that $\mathcal{C}/\mathcal{D} = \ker(i^{\mathsf{R}})$.

Construction 3.5. Pr^{L} has natural tensor product called Lurie Tensor product. Pr^{L} has symm monoidal structure defined by $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^{\mathsf{L}}$, then $\mathcal{C} \otimes \mathcal{D} \in \mathrm{Pr}^{\mathsf{L}}$ is determined by

$$\mathrm{Fun}^{\mathsf{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) = \mathrm{Fun}^{\mathrm{bicont}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \tag{11}$$

Remark 3.6.

- The symmetric monoidal structure is closed and the internal hom is $\mathrm{Fun}^{\mathsf{L}}(\mathcal{C}, \mathcal{D})$.
- In particular we have an explicit description

$$\mathcal{C} \otimes \mathcal{D} = \mathrm{Fun}^{\mathsf{L}}(\mathcal{C}, \mathcal{D}^{\mathrm{op}})^{\mathrm{op}}. \tag{12}$$

- Pr^{L} has a lot of idempotent algebra objects. $\mathrm{Sp} \in \mathrm{Pr}^{\mathsf{L}}$. An modules over Sp are presentable stable ∞-categories.

  Notation 3.7. $\mathrm{Pr}^{\mathsf{L}}_{\mathrm{st}} \subset \mathrm{Pr}^{\mathsf{L}}$ consists of presentable stable ∞-categories whose unit it Sp.

Definition 3.8. $\mathcal{C} \in \mathrm{Pr}^{\mathsf{L}}_{\mathrm{st}}$ is *dualizable* if it is dualizable in $\mathrm{Pr}^{\mathsf{L}}_{\mathrm{st}}$. Then $\mathcal{C}$ has a dual $\mathcal{C}^{\vee} = \mathrm{Fun}^{\mathsf{L}}(\mathcal{C}, \mathrm{Sp})$ with the maps

1. $\mathrm{ev} : \mathcal{C} \otimes \mathcal{C}^{\vee} \to \mathrm{Sp}$
2. $\mathrm{coev} : \mathrm{Sp} \to \mathcal{C}^{\vee} \otimes \mathcal{C}$

such that the duality conditions are satisfied.

Example 3.9.

1. $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0)$ and $\mathcal{C}_0 \in \mathrm{Cat}^{\mathrm{perf}}$, then $\mathcal{C}$ is dualizable.
2. $\mathcal{C} = \mathcal{D}(\mathsf{A})$ and $\mathcal{C}^{\vee} = \mathsf{D}(\mathsf{A}^{\mathrm{op}})$, The evaluation map is taking the tensor and the coevulation map is given by the diagonal over $\mathsf{A} \otimes \mathsf{A}^{\mathrm{op}}$.

Definition 3.10. Let $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ be the ∞-category

1. Objects are dualizable ∞-categories.
2. Morphisms are continuous functors whose right adjoint is also continuous (called *strongly continuous*).

Example 3.11.

$$\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}} \tag{13}$$

is a functor. This means that the map between Ind categories admits a right adjoint which is also strongly continuous. Also a continuous functors of compactly generated infinity categories is strongly continuous iff it takes $\mathcal{C}_0 \to \mathcal{D}_0$. This means that Ind functor is fully faithful.

Theorem 3.12. *Let $\mathcal{C}$ be a presentable stable ∞-category. TFAE:*

1. *$\mathcal{C}$ is dualizable*
2. *$\mathcal{C}$ is a retract of a compactly generated category in* $\mathrm{Pr}^{\mathsf{L}}_{\mathrm{st}}$
3. *The colimit functor*

$$\mathrm{colim} : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C} \tag{14}$$

   admits a left adjoint $\hat{\mathsf{Y}} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$
4. *$\mathcal{C}$ is generated by compactly exhaustible objects (to define).*
5. *There exists compactly generated categories $\mathcal{A}, \mathcal{B} \in \mathrm{Pr}^{\mathsf{L}}$ and a strongly continuous localization functor :*

$$\mathsf{L} : \mathcal{A} \to \mathcal{B} \tag{15}$$

   such that $\mathcal{C} = \ker \mathsf{L}$.

6. *If $\mathcal{D} \subset \mathcal{D}'$ is a fully faithful incluson in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, then $\mathcal{C} \otimes \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}'$ is fully faithful.*

7. *$\mathcal{C}$ satisfies Grothendieck AB6 axioms i.e. let I be an indexing set, for each $i \in I$, suppose given a filtered category J_i and a functor $f_i : J_i \to \mathcal{C}$. Then*

$$\prod_{I} \varinjlim_{J_i} f_i \cong \varinjlim_{\prod J_i} \prod_{i \in i} f_i(-). \tag{16}$$

(this property is true in Anima and also in any compactly generated category).

8. *There exists a fully faithful strongly continuous map $i : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is compactly generated.*

Sketch of some implications:

1. **2** $\implies$ **1:** retracts of dualizable are dualizable.
2. **2** $\implies$ **3:** Check 3 for compactly generated. Argument of Lurie in SAG implies right adjointness.
3. **3** $\implies$ **8:** $\hat{Y} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ does the job.
4. **3** $\implies$ **5:** Same argument as before using compact objects.
5. **8** $\implies$ **2:** Let $i : \mathcal{C} \to \mathcal{D}$ be the embedding, then $i^R i = \mathrm{id}$ giving $\mathcal{C}$ is a retract of $\mathcal{D}$.
6. **1** $\implies$ **2:** By presentabilty, there exists a bousfield localization functor $\mathrm{Ind}(\mathcal{C}^{\kappa}) \to \mathcal{C}$.

   Claim 3.13. This localization has a continuous section.

   Proof of. claim. It is equivalent to show that $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathrm{Ind}(\mathcal{C})) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ is essentially surjective. By dualizibility, we get

   $$\mathcal{C}^{\vee} \otimes \mathrm{Ind}(\mathcal{C}^{\kappa}) \to \mathcal{C}^{\vee} \otimes \mathcal{C}$$

   is again a Bousfield localization as tensoring preserves it which preserves essential surjectivity.

   □

4 Lecture III : Dualizable categories continued.

We recall the equivalent implications of the theorem from the last lecture (Theorem 3.12) along with two additional equivalent definitions $3'$ and $5'$:

Theorem 4.1. *Let $\mathcal{C}$ be a presentable stable ∞-category. TFAE:*

1. *$\mathcal{C}$ is dualizable*
2. *$\mathcal{C}$ is a retract of a compactly generated category in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$*
3. *The colimit functor*

   $$\mathrm{colim} : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C} \tag{17}$$

   admits a left adjoint $\hat{Y} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$

3' *$\mathcal{C}$ is ω_1-compactly generated and $\varinjlim : \mathrm{Ind}(\mathcal{C}^{\omega_1}) \to \mathcal{C}$ has a left adjoint.*

4. *$\mathcal{C}$ is generated by compactly exhaustible objects (to define).*
5. *There exists compactly generated categories $\mathcal{A}, \mathcal{B} \in \mathrm{Pr}^{\mathrm{L}}$ and a strongly continuous localization functor :*

   $$\mathrm{L} : \mathcal{A} \to \mathcal{B} \tag{18}$$

   such that $\mathcal{C} = \ker \mathrm{L}$.

5' *$\mathcal{C} = \ker(\mathsf{D}(\mathsf{A}) \to \mathsf{D}(\mathsf{B}))$ where $\mathsf{A} \to \mathsf{B}$ is a homological epimorphism of E_1-rings (i.e. $\mathsf{B} \otimes_{\mathsf{A}} \mathsf{B} \cong \mathsf{B}$).*

6. *If $\mathcal{D} \subset \mathcal{D}'$ is a fully faithful incluson in $\mathrm{Pr}^{\mathsf{L}}_{\mathrm{st}}$, then $\mathcal{C} \otimes \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}'$ is fully faithful.*

7. *$\mathcal{C}$ satisfies Grothendieck AB6 axioms i.e. let I be an indexing set, for each $\mathsf{i} \in \mathrm{I}$, suppose given a filtered category J_i and a functor $\mathsf{f}_\mathsf{i} : \mathsf{J}_\mathsf{i} \to \mathcal{C}$. Then*

$$\prod_{\mathrm{I}} \varinjlim_{\mathsf{J}_\mathsf{i}} \mathsf{f}_\mathsf{i} \cong \varinjlim_{\prod \mathsf{J}_\mathsf{i}} \prod_{\mathsf{i} \in \mathsf{i}} \mathsf{f}_\mathsf{i}(-). \tag{19}$$

(this property is true in Anima and also in any compactly generated category).

8. *There exists a fully faithful stronguly continuous map $\mathsf{i} : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is compactly generated.*

Back to proving the implications :

- AB6 is equivalent to the fact that the colimit functor $\varinjlim : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ preserves all products.
- For proving $3'$, we start by proving the following proposition:

Proposition 4.2. *If $\mathcal{C}$ is dualizable, then $\mathcal{C}$ is ω_1-compactly generated.*

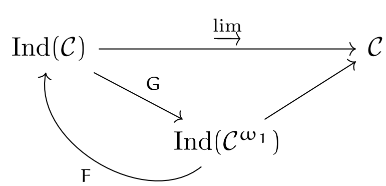
Proof. Use Criterion 5, $\mathcal{C} \cong \ker(\mathsf{L})$ where L is strongly continuous and is a morphism between compactly generated ∞-categories.

Claim 4.3. This forces $\mathcal{C}$ to be ω_1-compactly generated.

proof of the claim. An object of $\mathcal{C}$ is an ind-object $\{\mathsf{X}_\mathsf{i}\}_{\mathsf{i} \in \mathrm{I}}$ where $\mathsf{X}_\mathsf{i} \in \mathcal{A}$ such that $\{\mathsf{L}\mathsf{X}_\mathsf{i}\} \in \mathsf{B}$ has a zero colimit. This is an ind object in B^ω, the transition maps are everntually zero. We can write I to be the union of countable subsets where this holds. □

□

Proof of $3'$: If $\mathcal{C}$ is ω_1-cpt gen, then the colimit functor factors through $\mathrm{Ind}(\mathcal{C}^{\omega_1})$ which is a right adjoint G. G is essentially surjective and hence by diagram chasing below (also using the fact that $\mathcal{C}^{\omega_1}$ is small) we see that it preserves products.



(20)

Compact morphisms

Let $\mathcal{C} \in \mathrm{Pr}^{\mathsf{L}}_{\mathrm{st}}$

Definition 4.4. A morphism $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ in $\mathcal{C}$ is said to be *compact* if for every ind-object $\{\mathsf{Z}_\mathsf{i}\}_{\mathsf{i}\ in\mathrm{I}}$ such that $\varinjlim \mathsf{Z}_\mathsf{i} = 0$, then

$$\varinjlim_{\mathsf{i}} \mathrm{Hom}_{\mathcal{C}}(\mathsf{Y}, \mathsf{Z}_\mathsf{i}) \to \varinjlim_{\mathsf{i}} \mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Z}_\mathsf{i}) \tag{21}$$

is zero on π_*.

Remark 4.5. 1. X is compact iff id_X is a compact morphism.

2. Compact morphisms form a two-sided ideal.
3. A morphism factoring through a compact object is compact. The converse is true when $\mathcal{C}$ is compactly generated.

Definition 4.6. An object $X \in \mathcal{C}$ is *compactly exhaustible* iff it is a sequential colimit along compact maps.

Theorem 4.7 (Criterion 4 of Theorem 4.1)**.** *A presentable stable ∞-category if dualizable iff $\mathcal{C}$ is generated as a localizing subcategory by compactly exhaustible objects.*

Proof. If X is a sequential colimit of the form:

$$X = \varinjlim(Z_1 \to Z_2 \to \cdots) \tag{22}$$

where $Z_i \to Z_{i+1}$ are compact, then

$$\hat{Y}(X) = "\varinjlim"(Z_1 \to Z_2 \cdots).$$

Then we need to show that for every Ind system $"\varinjlim"W_j$, we need to show that :

$$\mathrm{Hom}_{\mathcal{C}}(X, \varinjlim W_j) = \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}("\varinjlim"Z_i, "\varinjlim"W_j). \tag{23}$$

We do in the following steps:

1. At first, we see that it is enough to show for the case where $\varinjlim_j W_j = 0$.
2. In that case we see that :

$$\begin{aligned} &\varinjlim_i \varinjlim \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(Y_i, "\varinjlim"W_j) \\ &\cong \varinjlim_i(\varinjlim_j \mathrm{Hom}_{\mathcal{C}}(Z_i, W_j)) \\ &\cong 0 \end{aligned}$$

   where the last line is zero as the transition maps $Z_i \to Z_{i+1}$ are compact.

Let us prove the other direction. If $\mathcal{C}$ is dualizalbe, then let $i : \mathcal{C} \to \mathcal{D}$ be fully faithful and strongly continuous with $\mathcal{D}$ compactly generated. $\mathcal{C}$ is ω_1-compactly generated. Let $Z \in \mathcal{C}$ be ω_1-compact.

Claim 4.8. Z is compactly exhaustible.

Proof of the Claim. $iZ \in \mathcal{D}$ is ω_1-compact, then

$$iZ = \varinjlim\{X_0 \to X_1 \cdots\} \tag{24}$$

where the morphisms and elements are in $\mathcal{D}$. Now remember that i admits a right adjoint i^R giving us the following :

$$Z = i^R iZ = \varinjlim(i^R x_0 \to i^R x_1 \cdots) \tag{25}$$

In order to show that the transition maps are compact, ETS that $ii^R x_m \to iii^R x_n$ compact for $n >> m$. Consider the following square :

$$\begin{array}{ccc} ii^R x_m & \longrightarrow & ii^R x_n \\ \downarrow & \overset{f}{\nearrow} & \downarrow \\ x_m & & x_n \end{array} \tag{26}$$

For n sufficiently large gives us the existence of the dotted arrow making the bottom triangle commute and hence the top triangle commute and as the top horizontal arrow actors through a compact object, this implies that desired arrow is compact. □

□

Proposition 4.9. *Let $\mathcal{C}, \mathcal{D}$ be dualizable. For a continuous functor $F : \mathcal{C} \to \mathcal{D}$, TFAE :*

1. F *is strongly continuous,*
2. F *commutes with* $\hat{Y}$.
3. F *preserves compact morphisms.*
4. F *is retract of* $\mathrm{Ind}(\mathcal{A}_0) \to \mathrm{Ind}(\mathcal{B}_0)$ *preserving compact objects.*

Let $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ *be the* ∞*-cat of dualizable cats and strongly continuous functors.*

Example 4.10. Some examples on dualizable categories related to homological epimorphisms are:

1. A morphism $A \to B$ is a map of $\mathbb{E}_1$-rings. This is a homological epimorphism if $D(B) \to D(A)$ is fully faithful. The kernel of extension of scalars is dualizable.
2. In the context of Almost ring theory (Faltings, Gabber-Romero). Let R be a commutative ring and $I \subset R$ ideal such that :
   - I is flat,
   - $I^2 = I$

   Then $R \to R/I$ is a homological epimorphism.
3. A, B perfect $\mathbf{F}_p$-algebras and $A \to B$ is a surjection, then it is a homological epimorphism.
4. Let X be a compact hausdorff space, then $\mathrm{Shv}(X, \mathrm{Sp})$ is dualizable. Here $\mathrm{Shv}(X, \mathrm{Sp}) \subset \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathrm{Sp})$ satisfying the sheaf condition:
   - $\mathcal{F}(\phi) = 0$
   - If $U_1, U_2 \subset X$, then the square

$$\begin{array}{ccc} \mathcal{F}(U_1 \cup U_2) & \longrightarrow & \mathcal{F}(U_1) \\ \downarrow & & \downarrow \\ \mathcal{F}(U_2) & \longrightarrow & \mathcal{F}(U_1 \cap U_2) \end{array} \tag{27}$$

   is a pullback square.
   - Given a filered union U_i of open sets

$$\mathcal{F}(\cup U_i) = \varprojlim \mathcal{F}(U_i).$$

5 Lecture IV: $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ and localizing invariants.

The following is a technical statement related to $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$:

Theorem 5.1 (Ramzi)**.** $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ *is* ω_1*-presentable.*

Remark 5.2.
- One can think of a dualizable category $\mathcal{C}$ is determined by $\mathcal{C}^{\omega_1}$. $\mathcal{C}^{\omega_1}$ is a stable ∞-category with countable colimits.
- $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ is complete nd cocomplete.

Example 5.3. The functor

$$\mathrm{Ind} : \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \tag{28}$$

is fully faithful and it peserves colimits. This is because the existence of the right adjoint $\mathcal{C} \mapsto \mathcal{C}^{\omega}$

Theorem 5.4. *The functor*

$$\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathrm{Pr}_{\mathrm{st},\omega_1}^{\mathsf{L}} \tag{29}$$

given by

$$\mathcal{C} \longmapsto \mathcal{C} \tag{30}$$

creates colimits. Here the right hand side is the ∞*-category of presentable stable* ∞*-categories which are* ω_1*-compactly generated and morphisms are continuous functors preserving* ω_1*-compact objects.*

Remark 5.5. The above functor has a right adjoint

$$\mathcal{D} \longmapsto \mathrm{Ind}(\mathcal{D}^{\omega_1}) \tag{31}$$

Proposition 5.6. *Given a diagram of dualizable cats and strongly continuous functors, then* $\varinjlim \mathcal{C}_i$ *is dualizable (here the limit is in* Pr^{L}*).*

Proof. We have that the colimit computation is same as limit in $\mathrm{Pr}_{\mathrm{st}}^{\mathsf{R}}$. Notice that AB6 holds in any term and in the transition functors hence it holds in the limit. □

Remark 5.7. A similar argument proves that if $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ is a strongly continuous functor in $\mathrm{Pr}_{\mathrm{st}}^{\mathsf{L}}$ and if $\mathcal{C}$ is dualizable. Then the localizing subcategory generated by $\mathsf{F}(\mathcal{C})$ is dualizable and the includion of this category in $\mathcal{D}$ is strongly continuous.

Example 5.8. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$. Let $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ be a continuous functor. Form $\mathcal{E} = \mathcal{C} \oplus_{\mathsf{F}} \mathcal{D}$ consists of $x \in \mathcal{C}, y \in \mathcal{D}$ and $\phi : x \to \mathsf{F}(y)$.

Claim 5.9.

- $\mathcal{E} = \langle \mathcal{C}, \mathcal{D} \rangle$
- $\mathcal{E} \in \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$
- $i_1 : \mathcal{C} \to \mathcal{E}$ and $i_2 : \mathcal{D} \to \mathcal{E}$ are strongly continuous.
- A functor $\mathcal{E}' \to \mathcal{E}$ is strongly continuous iff its projections to $\mathcal{C}, \mathcal{D}$ are strongly continuous.
- Any strongly continuous s.o.d arises in this way where $\mathsf{F} = i_1^{\mathsf{R}} i_2$

Definition 5.10. A *short exact sequence* in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ is a sequence

$$0 \to \mathcal{C} \to \mathcal{D} \to \mathcal{C}/\mathcal{D} \to 0 \tag{32}$$

where

- $\mathcal{D} \to \mathcal{C}$ is fully faithful
- $\mathcal{C}/\mathcal{D} := \mathcal{C} \coprod_{\mathcal{D}} 0$.

Remark 5.11.

1. $\mathcal{D} := \ker(\mathcal{C} \to \mathcal{C}/\mathcal{D}) = \ker^{\mathrm{dual}}(\mathcal{C} \to \mathcal{C}\mathcal{D})$.
2. Ind completion of Karoubi sequence is a s.e.s
3. Any $\mathcal{C} \in \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ fits into a SES

$$0 \to \mathcal{C} \to \mathcal{C}' \to \mathcal{C}'/\mathcal{C} \to 0 \tag{33}$$

where $\mathcal{C}'.\mathcal{C}'/\mathcal{C}$ are compactly generated. Indeed this choice is functorial namely : $\mathcal{C}' = \mathrm{Ind}(\mathcal{C}^{\omega_1})$ and $\hat{y} : \mathcal{C} \xrightarrow{\subseteq} \mathrm{Ind}(\mathcal{C}^{\omega_1})$.

We see that $\mathcal{C}'/\mathcal{C}$ are all ind objects in $\mathcal{C}^{\omega_1}$ whose colimits are zero in $\mathcal{C}$.

Definition 5.12. We define

$$\mathrm{Calk}^{\mathrm{cont}}_{\omega_1}(\mathcal{C}) = (\mathrm{Ind}(\mathcal{C}^{\omega_1}/\mathcal{C})^{\omega}. \tag{34}$$

Have an exact functor $\mathcal{C}^{\omega_1} \to \mathrm{Calk}^{\mathrm{cont}}_{\omega_1}(\mathcal{C})$. If $x, y \in \mathcal{C}^{\omega_1}$, then Hom space in Calkin continuous category is given between two objects a, b as

$$\mathrm{Hom}_{\mathcal{C}}(a, b)/\,\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(x(a), \hat{y}(b)). \tag{35}$$

Proposition 5.13. *If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A} \to 0$, then the lower arrow in the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{B}/\mathcal{A} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A}' & \longrightarrow & \mathcal{B} \cup_{\mathcal{A}} \mathcal{A}' & \longrightarrow & \mathcal{B}/\mathcal{A} & \longrightarrow & 0 \end{array} \tag{36}$$

Proof. $\mathcal{A}' \to \mathcal{B} \cup_{\mathcal{A}} \mathcal{A}'$ is fully faithful. because pullback of colocalization is again on. □

Proposition 5.14. *If $\mathcal{C} \to \mathcal{B}/\mathcal{A}$ is strongly continuous, then $0 \to \mathcal{A} \to \mathcal{B} \times_{\mathcal{B}/\mathcal{A}} \mathcal{C} \to \mathcal{C} \to 0$ is again a s.e.s*

Proof. We need to show that $\mathcal{B} \times_{\mathcal{B}/\mathcal{A}} \mathcal{C}$is dualizable. We see that $\mathcal{B} \times_{\mathcal{B}/\mathcal{A}} \mathcal{C}$ embeds into the laxpullback of $\mathcal{B}$ and $\mathcal{C}$ is storngly continuous and in fact it is kernel of localziaiton to $\mathcal{B}/\mathcal{A}$. □

Theorem 5.15 (Effimov)**.** *There is a one-to one correspondence between localizing invariants on* $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ *and the localizing invariant on* $\mathrm{Cat}^{\mathrm{perf}}$*.*

In particular, given F a localizing invariant on $\mathrm{Cat}^{\mathrm{perf}}$*, we can define a localizing invariant on* $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ *denoted by F^{cont} where* $F^{\mathrm{cont}}(\mathcal{C}) = \Omega F(\mathrm{Calk}^{\mathrm{cont}}_{\omega_1}(\mathcal{C}))$

Theorem 5.16 (Tamme, Effimov)**.** *If*

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{F'} & \mathcal{A}_2 \\ \downarrow & & \downarrow \\ \mathcal{A}_3 & \xrightarrow{F} & \mathcal{A}_4 \end{array} \tag{37}$$

is a pullback square in $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$*. Then any localizing invariant will take this pullback square to a pullback square.*

6 Lecture V : Dualizable inverse limits.

Let $I \to \mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ sending $i \mapsto \mathcal{C}_i$ be a functor. We would like to compute the limit of this functor in the dualizable world. Abstractly, there is a continuous functor

$$\varprojlim^{\mathrm{dual}} \mathcal{C}_i \to \varprojlim^{\mathrm{Pr}^L} \mathcal{C}_i \tag{38}$$

which is not an equivalence.

Example 6.1.

- The limit may not be dualizable. Indeed the kernel of the map

$$\mathcal{D}(k[x, y]) \to \mathcal{D}(k[x^{\pm}]) \tag{39}$$

induced from the map of rings is not dualizable. This is because the kernel of this map is $\mathcal{D}(B)$ where Spec B is not convex and hence it is not dualizable (Effimov's characterization of rings R such that $\mathcal{D}(R)$ is dualizable).

- The limit is dualizable, but the maps from the limit to its components is not strongly continuous And example is kernel of the map $\mathcal{D}(\mathbf{Z}_p) \to \mathcal{D}(\mathbf{F}_p)$ which is $\mathcal{D}(\mathbf{Q}_p)$ but $\mathbf{Q}_p$ is locally compact and not compact thus the transition maps are not strongly continuous.
- Even if both exists, the limit in computed in Pr^L is different. An example is the map :

$$\varprojlim_i^{\mathrm{dual}} \mathcal{D}(\mathbf{Z}/p^n) \to \varprojlim_i \mathcal{D}(\mathbf{Z}/p^n) \cong \mathcal{D}(\mathbf{Z}_p)_p^\vee \tag{40}$$

  is not an equivalence where the L.H.S is the category $\mathrm{Nuc}(\mathbf{Z}_p)$ defined by Clausen-Scholze in Condensed math.

Remark 6.2. One of the problem in computing limits is that the compact objects do not match up.

Let us analyze the example of dualizable kernel. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$

Claim 6.3. $\ker^{\mathrm{dual}}(F) \subset \ker(F)$ is the maximal subcategory such that

1. $\ker^{\mathrm{dual}}(F)$ is dualizable.
2. $\ker^{\mathrm{dual}}(F) \subset \mathcal{C}$ is strongly continuous.

Proof. We start with a lemma

Lemma 6.4. *Given a collection of strongly continuous inclusions satisfying the above conditions $\mathcal{C}'_\alpha \subset \mathcal{C}$, then $\langle \mathcal{C}_\alpha \rangle_\alpha \subset \mathcal{C}$ is strongly continuous.*

Proof of the lemma. Take $\prod_\alpha \mathcal{C}_\alpha \to \mathcal{C}$ and consider its image. □

Using this lemma and the fact for any inclusion $\mathcal{C}' \subset \mathcal{C}$, there is a dualizable category $\mathcal{E}$ and strongly continuous functors $\mathcal{E} \to \mathcal{C}$ and $\mathcal{E} \to \mathcal{C}'$, the claim is proven. □

Example 6.5. Let

$$\begin{array}{ccc} & & \mathcal{A} \\ & & \downarrow F \\ \mathcal{B} & \xrightarrow{G} & \mathcal{C} \end{array} \tag{41}$$

be a diagram in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$. Then we have the following claim

Claim 6.6. $\mathcal{A} \times_{\mathcal{C}}^{\mathrm{dual}} \mathcal{B} := \ker^{\mathrm{dual}}(\mathcal{A} \times_{\mathcal{C}}^{\rightarrow} \mathcal{B} \to \mathcal{C})$ where the functors sends a pair $(X, Y, \phi : F(X) \to \mathcal{G}(Y))$ to $\mathrm{cofib}(\phi)$

Remark 6.7. Given a finite diagram in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ the functor Eq. (38) is fully faithful.

Now we move to case of countable colimits.

Definition 6.8. $\mathrm{Cat}_{\omega_1,\mathrm{st}}$ is the ∞-category of stable ∞-category with countable colimits and functors that preserve countable colimits,

Remark 6.9. $\mathrm{Cat}_{\omega_1,\mathrm{st}}$ is equivalent to $\mathrm{Pr}^L_{\omega_1,\mathrm{st}}$ by send $\mathcal{A} \to \mathrm{Ind}_{\omega_1}(\mathcal{A})$. Here the latter is the ∞-category of presentable stable ∞-category with are ω_1-compactly generated and functors that preserve ω_1-compact objects.

Theorem 6.10. *There is an adjuncton*

$$\mathrm{Cat}^{\mathrm{dual}} \leftrightarrows \mathrm{Cat}_{\omega_1,\mathrm{st}} \tag{42}$$

where

1. $\rightarrow$: $\mathcal{C} \mapsto \mathcal{C}^{\omega_1}$
2. $\leftarrow$: $\mathcal{A} \rightarrow \operatorname{Ind}(\mathcal{A})$

Remark 6.11. 1. $\mathcal{C} \xrightarrow{\hat{Y}} \operatorname{Ind}(\mathcal{C}^{\omega_1})$ is the unit of the adjunction.

2. Explicitly adjunction is given by $f : \mathcal{C}^{\omega_1} \rightarrow \mathcal{A}$, we get the functor $\mathcal{C} \xrightarrow{\hat{Y}} \operatorname{Ind}(\mathcal{C}^{\omega_1}) \xrightarrow{\operatorname{Ind}(F)} \operatorname{Ind}(\mathcal{A})$.

Proof. All it remains to check that the maps cosntructed in this way preserve (strongly) continuous conditions. For this we check the following: $\mathcal{C} \in \mathrm{Cat}^{\mathrm{dual}}, \mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}$. Then we have :

$$\operatorname{Fun}^{\mathrm{LL}}(\mathcal{C}, \operatorname{Ind}(\mathcal{D})) \cong \operatorname{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \tag{43}$$

The functors from both of sides are defined in the following manner:

1. $\longrightarrow$: Let $F : \mathcal{C} \rightarrow \operatorname{Ind}(\mathcal{D})$ be an element in LHS, then by definition is has a right adjoint continuous functor $F^R : \operatorname{Ind}(\mathcal{D}) \rightarrow \mathcal{C}$ which gives such a accessible limit preserving functor $\mathcal{D} \rightarrow \mathcal{C}$ via the inclusion $\mathcal{D} \hookrightarrow \operatorname{Ind}(\mathcal{D})$. Thus this defines a left adjoint map $\mathcal{C} \rightarrow \mathcal{D}$.
2. $\longleftarrow$: Let $G : \mathcal{C} \rightarrow \mathcal{D}$ which is a left adjoint map. This provides a limit presrving map $G^R : \mathcal{D} \rightarrow \mathcal{C}$ This induces a limit prserving functor $\operatorname{Ind}(\mathcal{D}) \rightarrow \operatorname{Ind}(\mathcal{C}) \rightarrow \mathcal{C}$. Taking right adjoints, we get the functor $\mathcal{C} \rightarrow \mathcal{D}$. Notice that the right adjoint map also preserves compact objects.

□

Remark 6.12. • The above works unstably.

• Ramzi proves that the above adjunction is comonadic.

Description of limit: Let $I \rightarrow \mathrm{Cat}^{\mathrm{dual}}$ given by $i \mapsto \mathcal{C}_i$, then we have a short exact sequence :

$$0 \rightarrow \mathcal{C}_i \xrightarrow{\hat{y}} \operatorname{Ind}(\mathcal{C}^{\omega_1}) \rightarrow \operatorname{Ind}(\operatorname{Calk}^{\mathrm{cont}}_{\omega_1}(\mathcal{C}_i)) \rightarrow 0 \tag{44}$$

Then we have

Theorem 6.13. *The limit can be computed as the following formula:*

$$\varprojlim_i{}^{\mathrm{dual}} \mathcal{C}_i = \ker^{\mathrm{dual}}(\operatorname{Ind}(\varprojlim_i \mathcal{C}_i^{\omega_1}) \rightarrow \operatorname{Ind}(\varprojlim_i \operatorname{Calk}^{\mathrm{cont}}_{\omega_1}(\mathcal{C}_i))). \tag{45}$$

Remark 6.14. In many cases, $\varprojlim_i \mathcal{C}_i^{\omega_1} \rightarrow \varprojlim_i \operatorname{Calk}^{\mathrm{cont}}_{\omega_1}(\mathcal{C}_i)$ is not a homological epimorphism. If it is a homological epimorphism, then $\ker^{\mathrm{dual}}$ in the limit computation can be replaced by ker.

Using the above formula and the fact that products of Karoubi sequences is Karoubi, we see that

$$\prod_i \operatorname{Ind}(\mathcal{A}_i) \cong \operatorname{Ind}(\prod_i \mathcal{A}_i). \tag{46}$$

Construction 6.15. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{dual}}$. Let S be a 2-sided ideal of compact morphisms. Suppose $S = S^2$.
Define $\mathcal{C}_S \subset \mathcal{C}$ to be the localizing subcategory generated by all sequential colimits where the transition maps are in S.

Proposition 6.16. *$\mathcal{C}_S$ is dualizable and $\mathcal{C}_S \subset \mathcal{C}$ is strongly continuous.*

Remark 6.17. $\mathcal{C}/\mathcal{C}_S$ is universal localizing subcategory where all morphisms in S go to zero.

Proof of the proposition. Need $\mathcal{C}_S$ to be generated by compactly exhaustible objects.

Observation Consider any seqeuential collection $i \to x_i$. Such a collection can be extended to the poset $\mathbf{Q}_{\geq 0}, \subset)$. Then

$$\varinjlim_i x_i = (\varinjlim_{(0,1)} x_i \to \varinjlim_{(2,3)} x_i \to \varinjlim_{(4.5)} x_i \cdots). \tag{47}$$

This shows that $\mathcal{C}_S$ be generated by compactly exhaustible objects. □

The above proposition gives an alternate description of the kernel for any morphism of dualizable categories.

Description of the kernel: Take S_0 to be thecollection of compact morphisms in $\mathcal{C}$ that map to zero under F. Take $S = S_0^\infty$ to te the collection of f that can be extended from $\{0 \to 1\}$ over $[0.1]_{\mathbf{Q}}$. Then we see that $\ker^{\text{dual}}(F) = \mathcal{C}_S$.

Remark 6.18. Using the description of the kernel and Theorem 6.13, we see that $\varprojlim^{\text{dual}} \mathcal{C}_i$ is generated by functors $f : (\mathbf{Q}, \subset) \to \varprojlim \mathcal{C}_i^{\omega_1}$ such that $r \leq s$ $f(r) \to f(s)$ is a compact morphisms for each $\mathcal{C}_{i_0}$ where $i_0 \in I$.