From infinite-dimensional Teichmüller theory to conformal field theory and back.

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Congratulations to Alan Carey on his 60th birthday.

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Conformal Field Theory (CFT):

- Special class of 2D quantum field theories.
- First mathematical definition (G. Segal, Kontevich $\approx$ 1986)
- Deeply connected to algebra, topology and analysis.

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- (∞-dim) Teichmüller space of Riemann surfaces
- Sewing=gluing=welding
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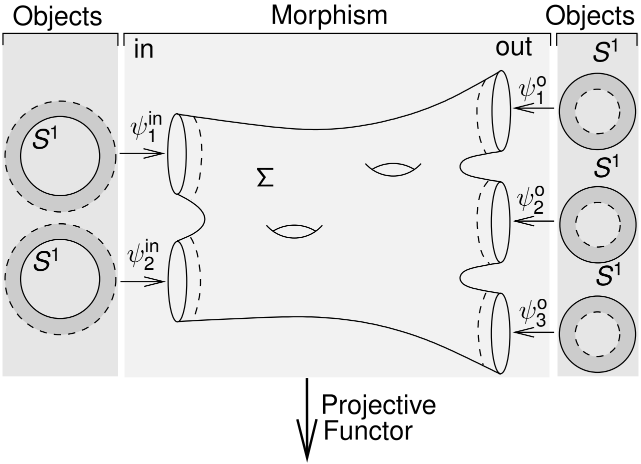
Our General Aim:

- Provide a natural **analytic** setting for the rigorous definition of CFT in higher genus. Definitions and Theorems.
- Use CFT ideas to prove new results in Teichmüller theory and geometric function theory.

Motivation/Application: Conformal Field Theory



$$H \otimes H \xrightarrow{A([\Sigma, \psi])} H \otimes H \otimes H$$

The construction of CFT using Vertex Operator Algebras is nearing completion.

In the Conformal Field Theory (CFT) definition.

- Basic object: Rigged moduli space =
  { Riemann surf. with parametrized boundary } / conf. equiv.
- Basic operation: Sewing surfaces.
- The sewing operation must be a holomorphic operation on the rigged moduli spaces.

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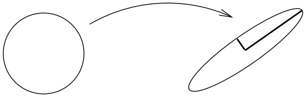
Some of our results (2006 - 2009): Teichmüller theory $\Longleftrightarrow$ CFT

- $\Longrightarrow$ The rigged moduli space can be obtained (as a complex Banach manifold) from the usual Teichmüller space.
- $\Longrightarrow$ The sewing operation is holomorphic.
- $\Longleftarrow$ A fiber structure and new coordinates for Teichmüller space.

Quasiconformal Maps

$f : \Omega \subset \mathbb{C} \to \mathbb{C}$. Homeomorphism. L^1_{loc} derivatives.

Jacobian(f) :



Circular Dilatation = major axis / minor axis.

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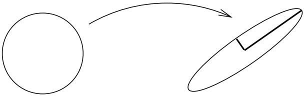
Jacobian(f) : 

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Definition

f is K-**quasiconformal** if its circular dilatation is globally bounded by K.

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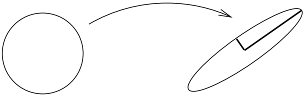

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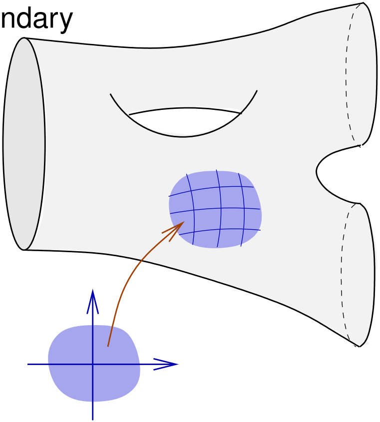
Definition

$h : S^1 \to S^1$ is **quasisymmetric** if and only if it can be extended to a quasiconformal mapping of the disk to itself.

Note: There are numerous equivalent definitions.

Basic Objects

- Riemann Surfaces with boundary



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- Riemann Surfaces with boundary
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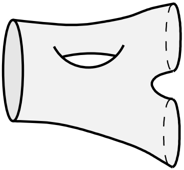

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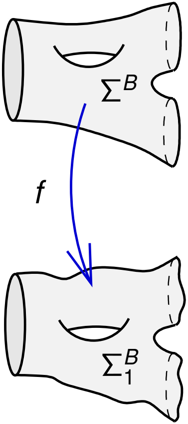
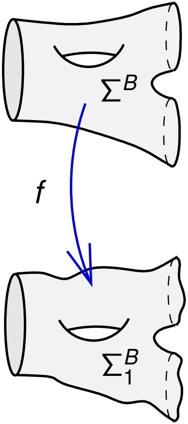
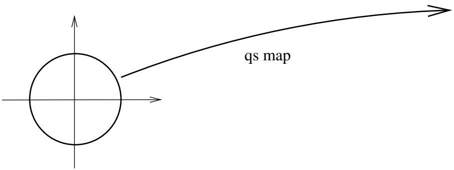


Basic Objects

- Riemann Surfaces with boundary
- Fix the genus and number of boundary components.
- Quasiconformal map
- Quasisymmetric boundary parametrization





Teichmüller Space $=$ space of Riemann surfaces

Fix a base Riemann surface Σ.
Given Σ_1 and quasiconformal $f : \Sigma \to \Sigma_1$, write (Σ, f, Σ_1).

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Definition (Teichmüller space:)

$T(\Sigma) = \{(\Sigma, f, \Sigma_1)\}/\sim$.

$$(\Sigma, f, \Sigma_1) \sim (\Sigma, g, \Sigma_2) \iff \exists \text{ conformal } \sigma : \Sigma_1 \to \Sigma_2 \text{ such that } g^{-1} \circ \sigma \circ f \approx \mathrm{id} \text{ (rel. boundary)}$$

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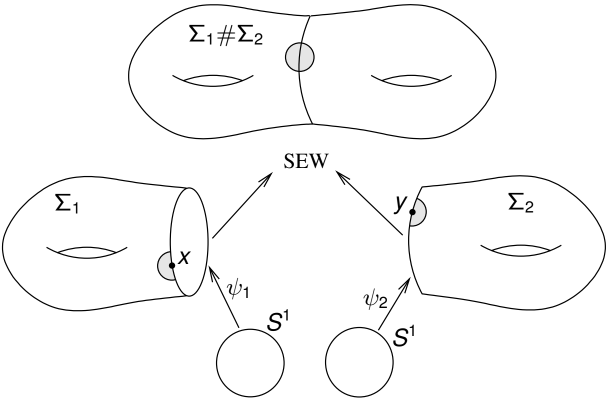
Classical Facts:

1. If Σ is closed (with punctures) then $T^P(\Sigma)$ is a finite-dimensional complex manifold.
2. If Σ is a surface with boundary then $T^B(\Sigma)$ is an ∞-dimensional complex manifold.
3. Moduli space $= T(\Sigma)/$ (Mapping Class Group).

Sewing using parametrizations

$$\Sigma_1 \# \Sigma_2 = (\Sigma_1 \sqcup \Sigma_2) / \left(\psi_1^{-1}(x) = \psi_2^{-1}(y)\right)$$



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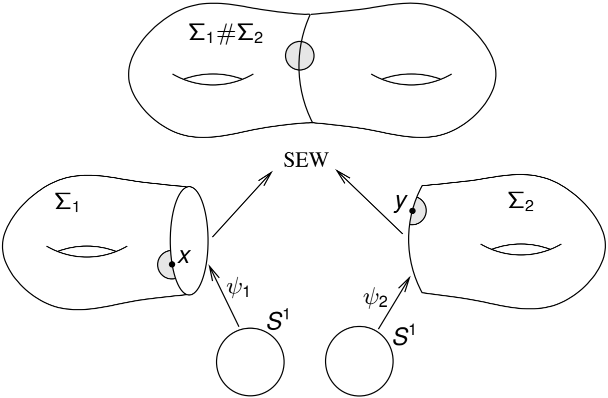


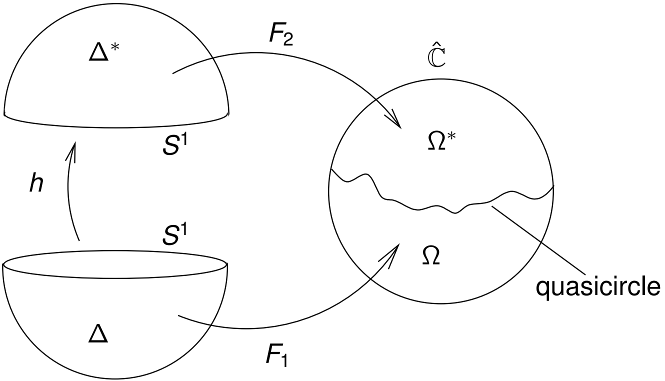
Note: If ψ_i are conformal then $\Sigma_1 \# \Sigma_2$ immediately becomes a Riemann surface. This is what was previously used in CFT.

Conformal Welding – a key classical fact

Δ – unit disk, $\Delta^* = \hat{\mathbb{C}} \setminus \bar{\Delta}$, $h : S^1 \to S^1$ (quasisymmetry)

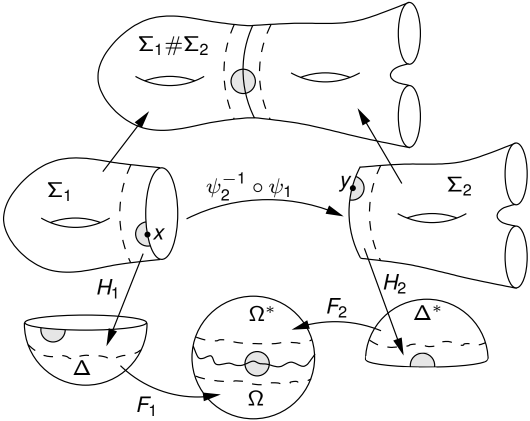
Theorem (conformal welding:)

There exists conformal maps F_1 and F_2 such that $F_2^{-1} \circ F_1 = h$ on S^1.



Quasisymmetric Sewing

ψ_1 and ψ_2 – quasisymmetric boundary parametrizations.
Define charts on $\Sigma_1 \# \Sigma_2$ by:



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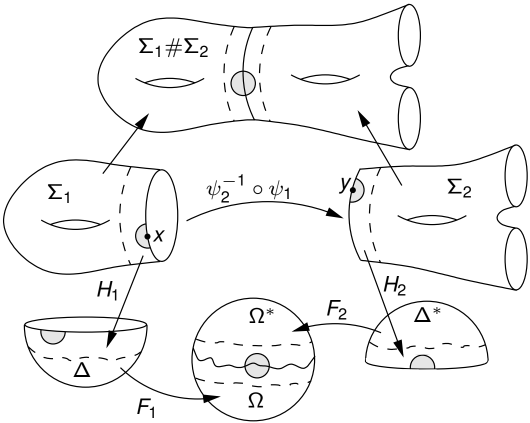


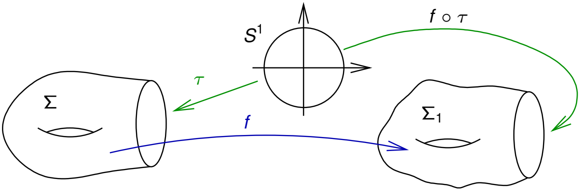
Proposition (RS 2006)

This gives the unique complex structure on $\Sigma_1 \# \Sigma_2$ which is compatible with Σ_1 and Σ_2.

Results: Teichmüller theory $\Longrightarrow$ CFT

Key idea:

Fix a QS parametrization of the base surface: $\tau : S^1 \to \partial\Sigma$.
Then $[\Sigma, f, \Sigma_1] \in T^B(\Sigma)$ **contains boundary parametrization data** for Σ_1 via $f \circ \tau$.



Theorem (R. - Schippers 2006)

(1) $T^B(\Sigma)$ / (discrete mapping class group) = rigged moduli space

(2) $T^B(\Sigma_1) \times T^B(\Sigma_2) \xrightarrow{sew} T^B(\Sigma_1 \# \Sigma_2)$ is holomorphic.

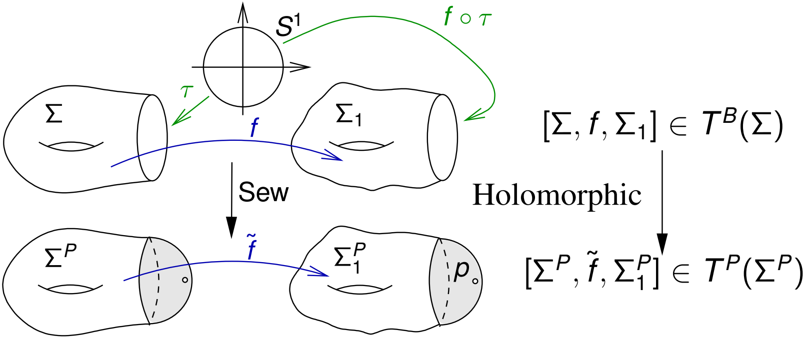
Proofs: (1) Technical. Use lambda lemma from complex dynamics etc.(2) easy

Cap Sewing: CFT $\Longrightarrow$ Teichmüller theory

Theorem (R. - Schippers 2009)

1. *T^B is a holomorphic fiber space over T^P.*
2. *The fibers are complex Banach manifolds:*

$\mathcal{O}_{qc}(\Sigma_1^P) = \{g : \mathbb{D} \to \Sigma_1^P \mid g \text{ is } 1-1, \textit{ holo., has qc ext., and } f(0) = p\}.$



Relation to universal Teichmüller space

Fiber locally modeled on:

$$\mathcal{O}_{\mathrm{qc}} = \{f\colon \mathbb{D} \to \mathbb{C} \mid f \text{ is one-to-one, holomorphic, has qc extension to } \mathbb{C}, \text{ and } f(0) = 0\}.$$

Universal Teichmüller space (non-standard normalization)

$$\mathcal{D} = \{f\colon \mathbb{D} \to \overline{\mathbb{C}} \mid f \text{ is one-to-one, holomorphic, has qc extension to } \overline{\mathbb{C}}, \text{ and } f(0) = 0, f'(0) = 1, f''(0) = 0\}$$

Teichmüller curve (model due(?) to L.-P. Teo, 2004).

$$\widetilde{\mathcal{D}} = \{f\colon \mathbb{D} \to \mathbb{C} \mid f \text{ is one-to-one, holomorphic, has qc extension to } \mathbb{C}, \text{ and } f(0) = 0, f'(0) = 1\}.$$

Complex structure on $\mathcal{O}_{\mathrm{qc}}$

$$A^2_\infty(\mathbb{D}) = \{\psi(z)\colon \mathbb{D} \to \mathbb{C} \mid \psi \text{ holomorphic, } ||\psi||_{2,\infty} = \sup_{z\in\mathbb{D}}(1-|z|^2)^2|\psi(z)| < \infty\}$$

$$A^1_\infty(\mathbb{D}) = \{\phi(z)\colon \mathbb{D} \to \mathbb{C} \mid \phi \text{ holomorphic, } ||\phi||_{1,\infty} = \sup_{z\in\mathbb{D}}(1-|z|^2)|\phi(z)| < \infty\}.$$

pre-Schwarzian: $\mathcal{A}(f) = \dfrac{f''}{f'}$ Schwarzian: $\mathcal{S}(f) = \dfrac{f'''}{f'} - \dfrac{3}{2}\left(\dfrac{f''}{f'}\right)^2$

Embeddings: $\mathcal{S}\colon \mathcal{D} \to A^2_\infty(\mathbb{D})$ and $\mathcal{A}\colon \widetilde{\mathcal{D}} \to A^1_\infty(\mathbb{D})$

The complex structures are compatible (L.-P. Teo 2004).

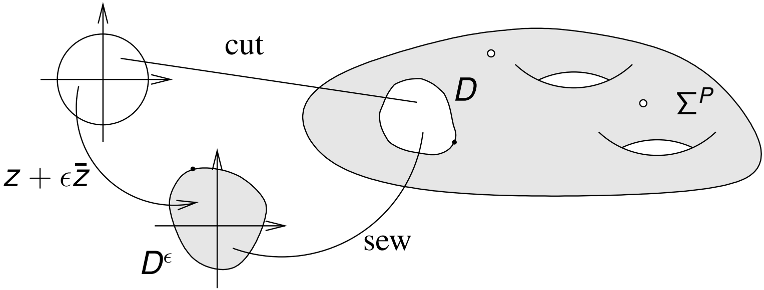
Corollary (R. - Schippers 2008)

The embedding

$$\begin{array}{rcl} \chi\colon \mathcal{O}_{qc} & \to & A^1_\infty(\mathbb{D}) \oplus \mathbb{C} \\ f & \mapsto & \big(\mathcal{A}(f), f'(0)\big). \end{array}$$

defines a compatible complex structure on $\mathcal{O}_{qc}$

Schiffer variation = coords for Teichmüller space



$\Sigma^\epsilon = (\Sigma^P \setminus D)\# D^\epsilon$. Let $N = 3g - 3 + n$.

Theorem (Gardiner 1975, Nag 1985)

Variation on any N disks, $(\epsilon_1, \ldots \epsilon_N) \mapsto \Sigma^{(\epsilon_1, \ldots \epsilon_N)}$, gives local holomorphic coordinates for the Teichmüller space of punctured surfaces.

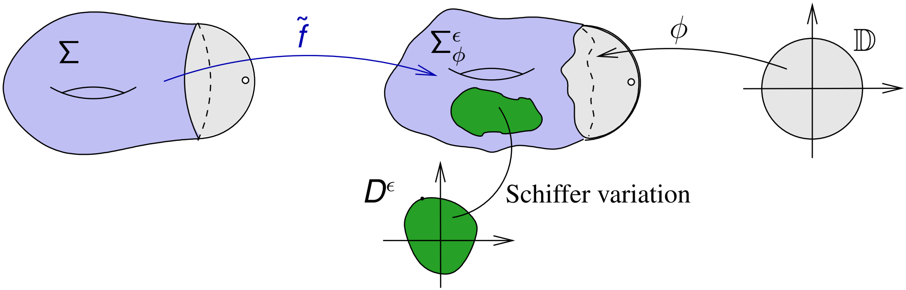
Fix $[\Sigma^P, f, \Sigma_1^P] \in T(\Sigma^P)$. Choose $\Omega \subset \mathbb{C}^N$ and N Schiffer variation disks on Σ_1^P. Let $U \subset \mathcal{O}_{\text{qc}}(\Sigma_*)$. **Let $\Sigma_\phi^\epsilon = \Sigma_1^\epsilon \setminus \phi(\mathbb{D})$.**

Theorem (R. - Schippers 2009)

The map

$$\Omega \times U \to T^B(\Sigma)$$
$$(\epsilon, \phi) \mapsto [\Sigma, \tilde{f}, \Sigma_\phi^\epsilon]$$

is a local holomorphic coordinate chart on $T^B(\Sigma)$



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