Traces on log-polyhomogeneous pseudodifferential operators

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Contents

Introduction

Here are some known uniqueness results for traces on certain algebras of polyhomogeneous (or classical) PDOs:

- The usual trace on smoothing operators. [Guillemin]
- The noncommutative residue on classical PDOs. [Wodzicki]
- The canonical trace on non integer order classical PDOs (follows from [Lesch]) and odd-class PDOs in odd dimension. [Maniccia-Schrohe-Seiler]

We derive from these results of uniqueness on classical PDOs, similar uniqueness results on subalgebras of log-polyhomogeneous PDOs. Precisely, let τ be a trace on a class of log-polyhomogeneous PDOs:

$$\bullet \quad \tau \quad \text{unique} \quad \text{on} \quad \text{classical} \quad \text{PDOs}$$

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This also holds for $\mathbb{Z}$-graded traces.

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Pseudodifferential operators

M : closed riemannian manifold of finite dimension n.
E : finite rank hermitian vector bundle over M.
All operators will act on sections of E.

We recall that a PDO A is polyhomogeneous (or classical) if locally its symbol $\sigma(A)(x,\xi)$ admits an asymptotic expansion in positively homogeneous components with decreasing degree of homogeneity:

$$\sigma(A)(x,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\, \sigma_{m-j}(A)(x,\xi).$$

m is the order of the PDO A.

A PDO L is log-polyhomogeneous if locally its symbol has the form

$$a_k(x,\xi) \ln^k |\xi| + a_{k-1}(x,\xi) \ln^{k-1} |\xi| + \cdots + a_1(x,\xi) \ln |\xi| + a_0(x,\xi).$$

$k \in \mathbb{N}$ and $a_0, \cdots, a_k$ are classical symbols.
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Odd-class operators

A classical PDO A with local symbol

$$\sigma(A)(x,\xi) \sim \sum_{j=0}^{\infty} \chi(\xi)\, \sigma_{m-j}(A)(x,\xi).$$

is *odd-class* if the positively homogeneous components of its symbol $\{\sigma_{m-j}(A) : j \in \mathbb{N}\}$ are simply homogeneous, i.e. they have the property

$$\sigma_{m-j}(A)(x,-\xi) = (-1)^{m-j} \sigma_{m-j}(A)(x,\xi)$$

A log-polyhomogeneous PDO L is odd-class if locally, all the classical symbols $a_0, \cdots, a_k$ arising in its symbol have the above property.

Usual trace

We denote by $C\ell$ the algebra generated by all classical PDOs.

A classical PDO A of order $< -n$ is trace-class. Its symbol $\sigma(A)(x, \cdot)$ at a point x lies in $L^1(\mathbb{R}^n)$ and the *trace* of A, which is finite, is given by

$$\mathrm{Tr}(A) := \int_M \left(\int_{T_x^* M} \mathrm{tr}_x \left(\sigma(A)(x, \xi) \right) \, đ\xi \right) dx.$$

Properties:

- The trace Tr is the unique trace on smoothing operators. [Guillemin]
- The trace Tr does not extend to a trace functional on the whole algebra $C\ell$. (Follows from [Wodzicki])

Noncommutative residue ([Guillemin] and [Wodzicki])

The noncommutative residue of A is defined by

$$\mathrm{res}(A) = \int_M \mathrm{res}_x(A)dx = \int_M \left(\int_{S_x^* M} \mathrm{tr}_x \left(\sigma_{-n}(A)(x,\xi)\right) đ\xi \right) dx.$$

Properties:

- $\mathrm{ord}(A) < -n \quad \mathrm{or} \quad \mathrm{ord}(A) \notin \mathbb{Z} \Rightarrow \mathrm{res}(A) = 0.$
- If the manifold M is connected with dimension >1, res is the unique trace defined on $C\ell$. [Wodzicki]

The residue density extends to log-polyhomogeneous PDOs. [Lesch]
For L of log-degree k with local symbol
$\sigma(L)(x,\xi) = a_k(x,\xi)\ln^k|\xi| + a_{k-1}(x,\xi)\ln^{k-1}|\xi| + \cdots + a_0(x,\xi)$,

$$\mathrm{res}_k(L) = (k+1)! \int_M \int_{S_x^* M} \mathrm{tr}_x((a_k)_{-n}(x,\xi))đ\xi dx.$$

Canonical trace ([Kontsevich] and [Vishik])

Let A be a classical PDO with non integer order. The canonical trace of A is defined by

$$\mathrm{TR}(A) := \int_M \mathrm{TR}_x(A)dx = \int_M ⨍_{T_x^*M} \mathrm{tr}_x\left(\sigma(A)(x,\xi)\right) đ\xi dx.$$

Properties:

- $\mathrm{TR}(A) = \mathrm{Tr}(A)$ if $\mathrm{ord}(A) < -n$.
- TR is the unique trace on classical PDOs with non integer order which extends the usual trace. [Maniccia-Schrohe-Seiler] (also follows from [Lesch])
- If the dimension of M is odd, TR is the unique trace on $C\ell_{odd}$. [Maniccia-Schrohe-Seiler]

Remark

The canonical trace TR *has been extended to log-polyhomogeneous PDOs with non integer order. [Lesch]*

Subalgebra $\mathcal{A}$ of $\mathcal{L}$

Let Q be an admissible classical PDO with positive order.
$\mathcal{L}$ denotes the algebra of log-polyhomogeneous PDOs.

Lemma

- $\mathcal{L} = \bigoplus_{k=0}^{\infty} C\ell \operatorname{Log}^k Q$ *so that* $\mathcal{L}$ *is a* $\mathbb{Z}$*-graded algebra.*
- $\mathcal{L}_{odd} = \bigoplus_{k=0}^{\infty} C\ell_{odd} \operatorname{Log}^k Q$ *with* Q *odd-class and even order.*

Definition

Let $\mathcal{A}$ *be a subalgebra of* $\mathcal{L}$ *which contains* $\operatorname{Log} Q$.

$$\mathcal{A} = \bigoplus_{k=0}^{+\infty} \mathcal{A}_{C\ell} \operatorname{Log}^k Q$$

where $\mathcal{A}_{C\ell}$ *is the subalgebra of classical PDOs of* $\mathcal{A}$.

- $\mathcal{R}$ *is the ideal of smoothing operators in* $\mathcal{A}$.
- $\mathcal{A}^k$ *is the space of operators in* $\mathcal{A}$ *of log degree* k.

Useful result

Assumption: there exists a unique non-trivial trace τ on $\mathcal{A}_{C\ell}$.

Lemma

There exists $P \in \mathcal{A}_{C\ell}$ with $\tau(P) \neq 0$ such that for $A \in \mathcal{A}^k$, there exists $Q_i \in \mathcal{A}^k$, $P_i \in \mathcal{A}_{C\ell}$ and complex scalars α_i such that

$$A = \sum_{i=1}^{M} [P_i, Q_i] + P\left(\alpha_0 + \alpha_1 \mathrm{Log} Q + \cdots + \alpha_k \mathrm{Log}^k Q\right).$$

Sketch of proof: By induction on the log-degree k of A.

- $k = 0$. $\mathcal{A}^0 = \mathcal{A}_{C\ell} \Rightarrow$ there exists $P \in \mathcal{A}_{C\ell}$ such that $\tau(P) = 1$ and
$$A = \sum_{i=1}^{M} [P_i, Q_i] + \tau(A)P.$$
- $k > 0$. $A = A_0 + A_1 \mathrm{Log} Q + \cdots + A_{k+1} \mathrm{Log}^k Q$.
$$A_k \mathrm{Log}^k Q = \left(\sum_{i=1}^{N} [P_i, Q_i] + \tau(A_k)P\right) \mathrm{Log}^k Q$$
$$= \sum_{i=1}^{N} [P_i, Q_i \mathrm{Log}^k Q] + \sum_{i=1}^{N} Q_i [\mathrm{Log}^k Q, P_i] + \tau(A_k) P \mathrm{Log}^k Q.$$

First result: Uniqueness of graded trace

Definition

On a $\mathbb{Z}$-graded algebra $\mathcal{B} = \bigoplus_{k \geq 0} \mathcal{B}^k$, a $\mathbb{Z}$-graded trace is a sequence $\tau^{gr} = (\tau_k)_{k \in \mathbb{N}}$ s. t. τ_k vanishes on $\bigoplus_{0 \leq j \leq k-1} \mathcal{B}^j$ and for $A \in \mathcal{B}^k, B \in \mathcal{B}^j$,

$$\tau_{k+j}[A, B] = 0.$$

Theorem 1

Let τ^{gr} be a $\mathbb{Z}$-graded trace on $\mathcal{A}$. Then τ_0 unique $\Rightarrow \tau^{gr}$ unique.

Sketch of proof: Let $A = A_0 + A_1 \mathrm{Log} Q + \cdots + A_k \mathrm{Log}^k Q$.

- If $\tau_{0|\mathcal{R}} \neq 0$, we choose P smoothing and so is $P\mathrm{Log}^k Q$. Then
$$\tau_k(A) = \alpha_k \tau_k(P\mathrm{Log}^k Q) = \alpha_k \mathrm{Tr}(P\mathrm{Log}^k Q).$$
τ_k is determined by its restriction to smoothing operators.
- If $\tau_{0|\mathcal{R}} = 0$, then τ_0 is proportional to res and we have
$$\tau_k(A) = \mathrm{res}_k(A) \tau_k(P\mathrm{Log}^k Q).$$
τ_k is unique and proportional to res_k.

Application of Theorem 1

Example

We consider $\mathcal{A} = \mathcal{L}$. It follows that $\mathcal{A}_{C\ell} = C\ell$.
The $\mathbb{Z}$-graded trace is given by the higher noncommutative residues

$$\tau^{gr} = (\mathrm{res}_k)_{k\in\mathbb{N}}$$

defined by Lesch. Since $\tau_0 = \mathrm{res}$, Theorem 1 gives back the uniqueness of this $\mathbb{Z}$-graded trace proved by Lesch.

Second result: uniqueness of trace

Theorem 2

Let τ be a trace on $\mathcal{A}$. Then $\tau_{|\mathcal{A}_{C\ell}}$ unique $\Rightarrow \tau$ unique.

Sketch of proof: Let $A = A_0 + A_1 \mathrm{Log} Q + \cdots + A_k \mathrm{Log}^k Q$.

- If $\tau_{|\mathcal{R}} \neq 0$, then we choose P smoothing. L_k is also smoothing and hence
$$\tau(A) = \mathrm{Tr}(L_k).$$
This implies the uniqueness of the trace τ on $\mathcal{A}$.
- If $\tau_{|\mathcal{R}} = 0$, then $(\mathrm{res}_k)_{k\in\mathbb{N}}$ is the unique $\mathbb{Z}$-graded trace on $\mathcal{A}$. Since $\mathrm{res}_{k+1}(A) = 0$, A is a sum of commutators $[P_i, Q_i]$ with $P_i \in \mathcal{A}_{C\ell}$ and $Q_i \in \mathcal{A}^{k+1}$. Hence $\tau(A) = 0$.

Application of Theorem 2

Example (2)

We consider $\mathcal{A} = \mathcal{L}_{odd}$ so that $\mathcal{A}_{C\ell} = C\ell_{odd}$.
We assume that M is odd-dimensional.

The canonical trace TR can be extended to $\mathcal{L}_{odd}$.
Recently Maniccia, Schrohe and Seiler proved that TR is the unique trace on $C\ell_{odd}$.

Theorem 2 gives the uniqueness of the canonical trace on $\mathcal{L}_{odd}$.