ALGEBRAIC K-THEORY AND ARITHMETIC

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NOTATION

F a number field

$\mathcal{O}_F$ its ring of integers

$v \in \operatorname{spec} \mathcal{O}_F$ a nonzero prime ideal

$k_v := \mathcal{O}_F / v$

$|\cdot|_v = q_v^{-\operatorname{ord}(\cdot)}$ where $q_v = |k_v|$

Let A be a commutative, unital ring. Let $P(A)$ be the category of finitely generated, projective A-modules

$$K_0(A) := \frac{\langle [P] \,;\, P \in P(A) \rangle}{\langle [P \oplus Q] - [P] - [Q] \,;\, P, Q \in P(A) \rangle}$$

$$K_1(A) := GL(A) / E(A)$$

$$K_2(A) := \mathrm{Ker}\,(\, St(A) \longrightarrow E(A))$$

where $St(A) = \varinjlim_n St_n(A)$

$$St_n(A) = \frac{\langle x_{ij}^{\lambda} \quad 1 \le i,j \le n \,,\ \lambda \in A \rangle}{\langle x_{ij}^{\lambda} x_{ij}^{\mu} x_{ij}^{-\lambda-\mu} \,,\ [x_{ij}^{\lambda} ; x_{jl}^{\mu}] x_{il}^{-\lambda\mu} \text{ for } i \neq l \quad [x_{ij}^{\lambda} ; x_{kl}^{\mu}] \text{ for } j \neq k \rangle}$$

See the Milnor's book : Intr. to Alg. K-theory

Th'm (see the Milnor's book)

(1) $K_0(\mathcal{O}_F) = \mathbb{Z} \oplus Cl(\mathcal{O}_F)$

(2) $K_1(\mathcal{O}_F) = \mathcal{O}_F^\times$

(3) $K_0(L) = \mathbb{Z}$ for any field L

(4) $K_1(L) = L^\times$ (-"-)

(5) $K_2(L) = L^\times \otimes L^\times / \langle x \otimes (1-x) ;\ x \neq 0, 1 \rangle$

$\uparrow$

Matsumoto theorem

There is the exact sequence defining ~~the~~ class group:

$$\begin{array}{ccccccccccc}
0 \to & \mathcal{O}_F^\times & \longrightarrow & F^\times & \xrightarrow{val} & \bigoplus_v \mathbb{Z} & \longrightarrow & Cl(\mathcal{O}_F) & \to 0 \\
& \| & & \| & & \| & & \wr\| & \\
0 \to & K_1(\mathcal{O}_F) & \to & K_1(F) & \xrightarrow{\partial} & \bigoplus_v K_0(k_v) & \to & \tilde{K}_0(\mathcal{O}_F). &
\end{array}$$

Corollary:

$$\operatorname{ord}_{s=-n} \zeta_F(s) = \dim_{\mathbb{Q}} K_{2n+1}(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where $\zeta_F(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s}$ for $\operatorname{Re} s > 1$

is the Dedekind zeta function

Th'm (Quillen) If $\mathbb{F}_q$ is a finite field with q-elements, then

$$K_0(\mathbb{F}_q) = \mathbb{Z}$$

$$K_{2n}(\mathbb{F}_q) = 0 \quad \text{for } n > 0$$

$$K_{2n-1}(\mathbb{F}_q) \cong \mathbb{Z}/q^n-1 \quad \text{for } n > 0$$

D. Quillen constructed the topological space $BQP(A)$ and defined his K-theory:

$$K_n^Q(A) := \pi_{n+1}(BQP(A))$$

<u>Th'm (Quillen)</u> $K_n^Q(A) = K_n(A)$ for all $0 \le n \le 2$.

<u>From now on we will write $K_n(A)$ instead of $K_n^Q(A)$</u>.

<u>Th'm (Quillen)</u> $K_n(\mathcal{O}_F)$ is a finitely generated abelian group. (4)

<u>Th'm (Borel)</u>

$$K_n(\mathcal{O}_F) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & n = 0 \\ \mathbb{Q}^{r_1+r_2-1} & n = 1 \\ 0 & n = \text{even},\ n > 0 \\ \mathbb{Q}^{r_1+r_2} & n \equiv 1 \bmod 4 \\ \mathbb{Q}^{r_2} & n \equiv 3 \bmod 4 \end{cases}$$

Th'm (Quillen) There is the following exact sequence:

$$\to K_n(\mathcal{O}_F) \to K_n(F) \xrightarrow{\partial} \bigoplus_v K_{n-1}(k_v) \to$$

$$\to K_{n-1}(\mathcal{O}_F) \to \cdots$$

$$\cdots$$

$$\to K_1(\mathcal{O}_F) \to K_1(F) \xrightarrow{\partial} \bigoplus_v K_0(k_v) \to$$

$$\to K_0(\mathcal{O}_F) \to K_0(F) \to 0$$

Th'm (Soulé, Quillen)

$$K_{2n+1}(\mathcal{O}_F) = K_{2n+1}(F) \quad \text{for } n>0$$

$$0 \to K_{2n}(\mathcal{O}_F) \to K_{2n}(F) \xrightarrow{\partial} \bigoplus_v K_{2n-1}(k_v)$$

for $n > 0$

Def. $D(n) := \operatorname{div} K_{2n}(F) =$

$$= \bigcap_{r \geq 0} K_{2n}(F)^r$$

For $n=1$ the group $D(1)$ was considered by Bass-Tate

For $n \geq 1$ the group $D(n)$ was considered by Banaszak

Note that $D(n) \subset K_{2n}(\mathcal{O}_F)$ so $D(n)$ is finite and

if $D(n) \neq 0$, then $D(n)$ is not a divisible group!

Conjectures (Quillen-Lichtenbaum)

There are natural isomorphisms

$$K_{2n}(\mathcal{O}_F) \otimes \mathbb{Z}_l \xrightarrow{\sim} H^2_{et}(\mathcal{O}_F[\tfrac{1}{l}]; \mathbb{Z}_l(n+1))$$

$$K_{2n+1}(\mathcal{O}_F) \otimes \mathbb{Z}_l \xrightarrow{\sim} H^1_{et}(\mathcal{O}_F[\tfrac{1}{l}]; \mathbb{Z}_l(n+1))$$

C. Soulé first constructed such maps for $l > n$ and then Dwyer - Friedlander constructed these map for all $l > 2$ extending the result of Soulé.

Dwyer and Friedlander constructed the Atiyah-Hirzebruch type spectral sequence

$$E_2^{p,-q} = H^p_{et}(\mathcal{O}_F[\tfrac{1}{l}]; \mathbb{Z}_l(\tfrac{q}{2})) \Rightarrow K^{et}_{q-p}(\mathcal{O}_F[\tfrac{1}{l}])$$

and surjective homomorphisms

$$K_n(\mathcal{O}_F) \otimes \mathbb{Z}_l \twoheadrightarrow K^{et}_n(\mathcal{O}_F[\tfrac{1}{l}])$$

for $l \geqslant 2$ and $n \geqslant 2$.

Th'm (G.B) There is the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to & K_{2n}(\mathcal{O}_F)_l & \to & K_{2n}(F)_l & \to & \bigoplus_v K_{2n-1}(k_v)_l & \to \\
& \downarrow & & \downarrow & & \downarrow \wr & \\
0 \to & K_{2n}^{et}(\mathcal{O}_F[\tfrac{1}{l}]) & \to & K_{2n}^{et}(F)_l & \to & \bigoplus_v K_{2n-1}^{et}(k_v) & \to
\end{array}$$

Th'm (G.B., M. Kolster) The middle vertical arrow in the diagram above induces natural isomorphism

$$D(n)_l \xrightarrow{\simeq} D^{et}(n) \quad \text{where}$$

$$D^{et}(n) := \operatorname{div} K_{2n}^{et}(F)_l$$

Th'm (G.B., P. Zelewski)

$$\varprojlim_k{}^1 K_n(F; \mathbb{Z}/l^k) \xrightarrow{\simeq} \varprojlim_k{}^1 K_n^{et}(F; \mathbb{Z}/l^k)$$

for any F, $l > 2$ and $n \geq 2$.

Observe that Quillen–Lichtenbaum conjecture can be reformulated as follows:

$$\varprojlim_{k} K_n(F;\mathbb{Z}/l^k) \xrightarrow[\boxed{\text{Q-L conj}}]{\overset{?}{\simeq}} \varprojlim_{k} K_n^{et}(F;\mathbb{Z}/l^k)$$

for any F, $l>2$, $n\geq 2$.

Th'm (G. B. P. Zelewski)

(1) $\varprojlim_{k}{}^1 K_{2n}(F;\mathbb{Z}/l^k) = 0$

(2) there is an exact sequence:

$$0 \to D(n)_l \to \varprojlim_{k}{}^1 K_{2n+1}(F;\mathbb{Z}/l^k) \to \varprojlim_{k}{}^1 \bigoplus_{v} K_{2n}(k_v;\mathbb{Z}/l^k) \to 0$$

Moreover $\varprojlim_{k}{}^1 \bigoplus_{v} K_{2n}(k_v;\mathbb{Z}/l^k)$ is a torsion free group.

If F is a totally real field then for $n>0$, $n=$ odd and $l>2$ the Quillen–Lichtenbaum conj. can be reformulated as follows

Conj. (Quillen–Lichtenbaum)

$$|\zeta_F(-n)|_l^{-1} = \frac{|K_{2n}(\mathcal{O}_F)|_l}{|K_{2n+1}(\mathcal{O}_F)_l|}$$

where $|X| :=$ number of elements of a finite set X.

However under the same assumptions we have:

Th'm (Wiles, consequence of the Main Conj. in Iwasawa theory)

$$|\zeta_F(-n)|_l^{-1} = \frac{|K_{2n}^{et}(\mathcal{O}_F[\frac{1}{l}])|}{|K_{2n+1}^{et}(\mathcal{O}_F[\frac{1}{l}])|}$$

Note that

$$K_{2n+1}^{et}(\mathcal{O}_F[\tfrac{1}{l}]) \xrightarrow{\simeq} H^1_{et}(\mathcal{O}_F[\tfrac{1}{l}]; \mathbb{Z}_l(n+1)) \xrightarrow{\simeq}$$

$$\xrightarrow{\simeq} H^0_{et}(\mathcal{O}_F[\tfrac{1}{l}]; \mathbb{Q}_l/\mathbb{Z}_l(n+1)) \quad \text{So}$$

$$|K_{2n+1}^{et}(\mathcal{O}_F[\tfrac{1}{l}])| = |w_{n+1}(F)|_l^{-1}$$

where $w_k(L) :=$ maximal m such that the exponent of $G(L(\mu_m)/L)$ divides k. E.g. $w_1(\mathbb{Q}) = 2$, $w_2(\mathbb{Q}) = 24$

$w_1(L) = \#$ of roots of unity in L.

Th'm (G.B., M. Kolster)

For F totally real, $n \geq 0$, n - odd

$l > 2$

$$|D(n)_l| = \frac{|w_{n+1}(F)\,\zeta_F(-n)|_l^{-1}}{\prod_{v|l} |w_n(F_v)|_l^{-1}}$$

Corollary. For $n > 0$, n = odd, $l > 2$

$$|D(n)_l| = |w_{n+1}(\mathbb{Q})\,\zeta_{\mathbb{Q}}(-n)|_l^{-1} =$$

$$= |K_{2n}^{et}(\mathbb{Z}[\tfrac{1}{l}])|$$

Th'm (Levine, Merkurjev, Suslin)

$$K_3(\mathcal{O}_F) \otimes \mathbb{Z}_l \xrightarrow{\sim} H^1_{et}(\mathcal{O}_F[\tfrac{1}{l}]; \mathbb{Z}_l(2))$$

Th'm (Tate)

$$K_2(\mathcal{O}_F) \otimes \mathbb{Z}_l \xrightarrow{\sim} H^2_{et}(\mathcal{O}_F[\tfrac{1}{l}]; \mathbb{Z}_l(2))$$

Hence <u>Quillen-Lichtenbaum conjecture</u> holds for $n=1$.

In particular if F is totally real then

$$|\zeta_F(-1)|_l^{-1} = \frac{|K_2(\mathcal{O}_F)_l|}{|K_3(\mathcal{O}_F)_l|}$$

Bloch – Kato conjecture

Let $\operatorname{char} L \neq l$. Then the natural homomorphism

$$K_n^M(L)/l^k \xrightarrow{\sim} H^n(G_L; \mathbb{Z}/l^k(n))$$

$$\{a_1, \dots, a_n\} \longmapsto a_1 \cup \dots \cup a_n$$

is an isomorphism.

<u>Th'm</u>. (Voevodsky – Rost – Weibel)
Bloch – Kato conjecture holds.

It is known that <u>Bloch–Kato conjecture</u> implies <u>Quillen–Lichtenbaum conjecture</u>. Hence the <u>Quillen – Lichtenbaum conjecture</u> holds for all $n \geq 1$.

Let $C := Cl(\mathbb{Q}(\mu_l))_l$. Let

$$\omega : G(\mathbb{Q}(\mu_l)/\mathbb{Q}) \longrightarrow (\mathbb{Z}/l)^\times$$

be the Teichmüller character

$$\zeta^\sigma = \zeta^{\omega(\sigma)} \quad \text{for } \sigma \in G(\mathbb{Q}(\mu_l)/\mathbb{Q})$$

One can make the decomposition

$$C = \bigoplus_{i=1}^{l-1} C^{[i]} \quad \text{where}$$

$$C^{[i]} = \{ c \in C \; ; \; \sigma c = \omega^i(\sigma) c \text{ for all } \sigma \in G(\mathbb{Q}(\mu_l)/\mathbb{Q})$$

<u>Conjecture (Kummer-Vandiver)</u>

$$C^{[i]} = 0 \quad \text{for all } i \text{ even}$$

Conjecture (Iwasawa)

$$C^{[i]} = \text{cyclic for all } i \text{ odd}$$

One can prove the following isomorphisms:

$$D(n)/l \xrightarrow{\sim} H^2_{et}(\mathbb{Z}[\tfrac{1}{l}]; \mathbb{Z}_l(n+1)) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} H^2_{et}(\mathbb{Z}[\tfrac{1}{l}]; \mathbb{Z}/l(n+1)) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} C^{[l-1-n]}/l$$

Hence we can restate the above conj. as follows:

<u>Conj. (Kummer-Vandiver)</u>

$D(n)_l = 0$ for all n even $2 \leq n \leq l-1$

<u>Conj. (Iwasawa)</u>

$D(n)_l$ = cyclic for all n odd $1 \leq n \leq l-2$

Proposition. If $l \to \infty$ then the number of eigenspaces $\mathbb{C}^{[i]}$ s.t. $\mathbb{C}^{[i]} = 0$ also goes to ∞.

proof. It follows from the reformulation of the Kummer–Vandiver's and Iwasawa conj. in terms of $D(n)$. $\square$

Proposition (Kurihara)

$$C^{[l-3]} = 0 \quad \text{for any } l > 2$$

proof. Since $K_4(\mathbb{Z}) = 0 \implies$

$D(2) = 0$. $\square$

Note that

$C^{[i]} = e_{\omega^i} C$ where

$$e_{\omega^i} = \frac{1}{l-1} \sum_{\sigma \in G(\mathbb{Q}(\mu_l)/\mathbb{Q})} \omega^i(\sigma)\sigma^{-1}.$$

Hence it is clear that $C^{[l-1]} = 0$.

Th'm (Soulé). For $l > v(n)$ we have $C^{[l-n]} = 0$

where

$$\log v(n) \leq n^{224 n^4}.$$

Let $f \geq 1$, $f \in \mathbb{Z}$.

Consider the partial zeta function

$$\zeta_f(a,s) = \sum_{\substack{k \geq 1 \\ k \equiv a \bmod f}} \frac{1}{k^s} \qquad \text{for } \operatorname{Re} s > 1.$$

$\zeta_f(a,s)$ can be analytically continued to the whole complex plane except $s=1$. For each $n \geq 0$ $\zeta_F(a,-n)$ is a rational number.

Let $F/\mathbb{Q}$ be an abelian extension with conductor f. It means that f is the smallest natural number such that $\mathbb{Q} \subset F \subset \mathbb{Q}(\mu_f)$

Coates and Sinnott generalized the classical Stickelberger element and defined the following elements in the group ring $\mathbb{Q}[G(F/\mathbb{Q})]$.

Def. (Coates – Sinnott)

$$\Theta_n := \Theta_n(b, f)$$

$$\Theta_n(b,f) := (b^{n+1} - (b,F)) \sum_{\substack{(a,f)=1 \\ 1 \le a < f}} \zeta_f(a,-n)\,(a,F)^{-1}$$

where (a, F) is the restriction of the automorphism

$$\sigma_a : \mathbb{Q}(\mu_f) \longrightarrow \mathbb{Q}(\mu_f)$$

$$\xi_f^{\sigma_a} = \xi_f^{a}$$

One can also write:

$$\Theta_n = \sum_{\substack{(a,f)=1 \\ 1 \le a < f}} \Delta_{n+1}(a,b,f)\,(a,F)^{-1}$$

where

$$\Delta_{n+1}(a,b,f) := b^{n+1}\zeta_f(a,-n) - \zeta_f(ab,-n)$$

Th'm. (Coates – Sinnott)

(1) $\Delta_{n+1}(a,b,f)$ are integers if $\left(b, \, w_{n+1}(\mathbb{Q}(\mu_f))\right) = 1$

(2) $\Delta_{n+1}(a,b,f) \equiv a^n b^n \Delta_1(a,b,f) \mod f_n$ where $f_n = f \cdot \prod_{p \mid f} p^{v_p(n)}$.

Conjecture (Coates – Sinnott)

For each positive b with $(b, w_{n+1}(\mathbb{Q}(\mu_f))) =$ Θ_n annihilates $K_{2n}(\mathcal{O}_F)$.

Th'm (Stickelberger) Θ_0 annihilates $\overline{Cl(\mathcal{O}_F) = K_0(\mathcal{O}_F)}$.

Th'm (Coates – Sinnott) Θ_1 annihilates $K_{2n}(\mathcal{O}_F)_l$ for any $l > 2$ ~~with~~ under the assumption that $(b \, ; \, |K_{2n}(\mathcal{O}_F)|) = 1$.

Th'm (G. B)

Θ_n annihilates $D(n)_l$ if $l \nmid n$

$n \cdot \Theta_n$ annihilates $D(n)_l$ if $l \mid n$

The proof of the above theorem was based on the construction of the <u>Stickelberger's "splitting" map</u>

Λ :

$$0 \to K_{2n}(\mathcal{O}_F)_l \to K_{2n}(F)_l \underset{\Lambda}{\overset{\partial}{\rightleftarrows}} \bigoplus_v K_{2n-1}(k_v) \to$$

with the property that

$$\partial \circ \Lambda = \text{multiplication by } \begin{cases} \Theta_n & l \nmid \\ n\Theta_n & l \mid \end{cases}$$

For any $l > 2$ and $k \geqslant 0$ there is the following exact sequence

$$0 \to K_{2n}(\mathcal{O}_F)[l^k] \to K_{2n}(F)[l^k] \underset{\Lambda}{\overset{\partial}{\rightleftarrows}} \bigoplus_v K_{2n-1}(k_v)[l^k] \to D(n)_l \to$$

Hence the above theorem follows.

It was observed by Sinnott that the classical Stickelberger's theorem is equivalent to the existence of the Stickelberger's splitting map Λ:

$$0 \to \mathcal{O}_F^\times \to F^\times \underset{\Lambda}{\overset{val}{\rightleftarrows}} \bigoplus_v \mathbb{Z} \to Cl(\mathcal{O}_F) \to 0$$

s.t. $val \circ \Lambda$ = multiplication by Θ_0.

Observe that

(1) $\Theta_n = (b^{n+1} - 1)\,\zeta_{\mathbb{Q}}(-n)$ for $F = \mathbb{Q}$

(2) $GCD\{b^{n+1} - 1 \,;\, b \text{ prime and } (b \,;\, |w_{n+1}(\mathbb{Q})|\,|K_{2n}(\mathbb{Z})|) = 1\} =$

$= w_{n+1}(\mathbb{Q})$

Corollary (G.B.) If l does not divide $n w_{n+1}(\mathbb{Q}) \zeta_{\mathbb{Q}}(-n)$ then the following exact sequence

$$0 \to K_{2n}(\mathbb{Z}) \to K_{2n}(\mathbb{Q}) \to \bigoplus_p K_{2n-1}(\mathbb{F}_p) \to$$

splits. Moreover if $l \mid n$ and l is regular then the above exact sequence also splits.

Examples

(1) For $n=3$, $w_4(\mathbb{Q}) \zeta_{\mathbb{Q}}(-3) = 2$.

Hence

$$K_6(\mathbb{Q}) \cong K_6(\mathbb{Z}) \oplus \bigoplus_p K_5(\mathbb{F}_p)$$ up to 2-torsion

(2) For $n=5$, $w_6(\mathbb{Q}) \zeta_{\mathbb{Q}}(-5) = -2$

Hence

$$K_{10}(\mathbb{Q}) \cong K_{10}(\mathbb{Z}) \oplus \bigoplus_p K_9(\mathbb{F}_p)$$

up to 2-torsion

(3) For $n = 11$, $w_{12}(\mathbb{Q})\zeta_{\mathbb{Q}}(-11) = 2 \cdot 691$

It follows from the joint work of G.B. with M. Kolster that

$D(n) \simeq \mathbb{Z}/691$ up to 2-torsion

Hence the exact sequence

$$0 \to K_{22}(\mathbb{Z})_l \to K_{22}(\mathbb{Q})_l \to \bigoplus_p K_{21}(\mathbb{F}_p)_l \to 0$$

splits for each prime $l > 2$ except $l = 691$.

Th'm (G.B.) The exact sequence

$$0 \to K_{2n}(\mathbb{Z})_l \to K_{2n}(\mathbb{Q})_l \to \bigoplus_p K_{2n-1}(\mathbb{F}_p)_l \to 0$$

splits iff $l \nmid w_{n+1}(\mathbb{Q})\zeta(-n)$.

Example. For $n = 67$, $37 \| w_{68}(\mathbb{Q})\zeta(-67)$

so $D(67) \simeq \mathbb{Z}/37$ and the exact sequence

$$0 \to K_{134}(\mathbb{Z})_{37} \to K_{134}(\mathbb{Q})_{37} \to \bigoplus_p K_{133}(\mathbb{F}_p)_{37} \to 0$$

does not split.

Th'm (Tate)

$$K_2(\mathbb{Q}) \simeq K_2(\mathbb{Z}) \oplus \bigoplus_p K_1(\mathbb{F}_p)$$

Let K be a totally real field and let F/K be an abelian extension. Let f be the conductor of F and let K_f/K be the ray class field extension.
Consider the partial zeta function

$$\zeta_f(a,s) = \sum_{c \equiv a \bmod f} \frac{1}{Nc^s} \quad \text{for } \operatorname{Re} s > 1$$

where a, c, f are ideals of $\mathcal{O}_K$.

Coates defined Stickelbergers elements in this case as follows:

Def. (Coates)

$$\Theta_n(b,f) = \left(Nb^{n+1} - (b,F)\right) \sum_{(a;f)=1} \zeta_f(a,-n)\,(a,F)^{-1}.$$

Th'm (Deligne - Ribet - Coates)

If $(b, w_{n+1}(F)) = 1$ then

$\Theta_n(b, f) \in \mathbb{Z}[G(F/K)]$.

Remark. By the work of Siegel

$\Theta_n(b, f) \in \mathbb{Q}[G(F/K)]$

Th'm (G.B, C. Popescu). Let

$\Theta_1(b, f_k)$ annihilates

$K_2(\mathcal{O}_{F_k})_\ell$ for all $k \geq 1$.

Then $\Theta_n(b, f)$ annihilates

$D_F(n)_\ell$ for all $n \geq 1$. where

$F_k = F(\mu_{\ell^k})$

Corollary (G.B, C. Popescu). Let F/K be a CM abelian extension.

If the Iwasawa μ-invariant for F and ℓ is zero then $\Theta_n(b, f)$ annihilates $D_F(n)_\ell$ for all $n \geq 1$.

Remarks

(1) $$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

B_n = Bernoulli numbers

$B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$

$B_n = 0$ for $n > 1$, n = odd

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \quad \text{for } n \geq 0$$

(2) $$\frac{t e^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

$B_n(x)$ Bernoulli polynomials

$$B_n(1-x) = (-1)^n B_n(x)$$

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}$$

$$\zeta_f(b,-n) = -f^n \frac{B_{n+1}\left(\frac{b}{f}\right)}{n+1} \quad \text{for } n \geq 0$$

$0 < b < f$

$$B_0(x) = 1$$

$$B_1(x) = x - \frac{1}{2}$$

$$B_2(x) = x^2 - x + \frac{1}{6}$$

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