Exercises for the workshop on dualisable categories and continuous K-theory

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These are exercises collected from the workshop on Dualisable Categories and Continuous K–theory held at the MPIM Bonn on 9-14 September, 2024. The following are some of the available resources on this new subject:

(1) Efimov – K–theory and localizing invariants of large categories

(2) Krause–Nikolaus–Pützstück – Lecture notes on sheaves on manifolds

(3) Ramzi – The formal theory of dualizable presentable ∞–categories

(4) Lehner – Exercises for Continuous K–theory

Furthermore, some resources on the basics of algebraic K–theory include the following:

(5) Hebestreit–Wagner – Lecture notes on algebraic and hermitian K–theory

(6) Winges – Lecture notes on localisation and devissage in algebraic K-theory

(7) Hilman–McCandless – Lecture notes for an introduction on algebraic K–theory

Comments, corrections, and suggestions are of course very welcome!

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1 Exercises from day 1

Exercise 1.1. Let $\mathcal{D} \xhookrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{C}/\mathcal{D}$ be a Karoubi sequence.

(1) Show that i admits a right adjoint if and only if p does.

(2) In either case, show that $\mathcal{D} \xleftarrow{i^R} \mathcal{C} \xhookleftarrow{p^R} \mathcal{C}/\mathcal{D}$ is another Karoubi sequence.

(3) In the situation of (1), show that $i(\mathcal{D})$ and $p^R(\mathcal{C}/\mathcal{D})$ generate $\mathcal{C}$ as a stable category.

Exercise 1.2. Show that a natural transformation $(-)^{\simeq} \to \Omega^\infty \mathrm{K}(-)$ (which is a map in $\mathrm{Fun}(\mathrm{Cat}^{\mathrm{perf}}, \mathcal{S})$) uniquely enhances to a natural transformation of $\mathbb{E}_\infty$–monoids. **Hint:** *use the universal property of* CMon *as a right adjoint.*

Exercise 1.3. Use the procedure described during Lecture 2 to show that there is an equivalence $\mathrm{colim}_X F \simeq \lim_X F \in \mathrm{Pr}^L$ for any functor $F\colon X \to \mathrm{Pr}^L$ where X is an anima/∞–groupoid/space.

Exercise 1.4. Explicitly work out the duality data needed to witness that $\mathrm{Ind}(\mathcal{C}_0)$ is dualisable for any $\mathcal{C}_0 \in \mathrm{Cat}^{\mathrm{perf}}$. Similarly, work out the duality data witnessing that $\mathcal{D}(A)$ is dualisable with dual $\mathcal{D}(A^{\mathrm{op}})$ for any ring A. **Hint:** *use the mapping spectrum functor* $\hom_{\mathcal{C}_0}\colon \mathcal{C}_0^{\mathrm{op}} \times \mathcal{C}_0 \to \mathrm{Sp}$ *for the first part.*

Exercise 1.5. Let $\mathcal{C}_0 \in \mathrm{Cat}^{\mathrm{perf}}$ and write $\mathcal{C} := \mathrm{Ind}(\mathcal{C}_0)$. Construct the left adjoint $\widehat{Y}\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ to the colimit functor $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$. **Hint:** *the functor* $\widehat{Y}$ *is given explicitly by* $\mathrm{Ind}(Y_0)$ *where* $Y_0\colon \mathcal{C}_0 \hookrightarrow \mathrm{Ind}(\mathcal{C}_0) = \mathcal{C}$ *is the Yoneda embedding.*

Exercise 1.6. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a closed symmetric monoidal category. Show that retracts of dualisable objects are dualisable.

Exercise 1.7. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a colimit preserving functor between presentable categories. Show the following:

(1) If F admits a filtered colimit preserving right adjoint, then F preserves compact objects.

(2) If $\mathcal{C}$ is compactly generated and F preserves compact objects, then the right adjoint of F preserves filtered colimits.

(3) If F is fully faithful and its right adjoint preserves filtered colimits, then F reflects compact objects (i.e. if $F(x) \in \mathcal{D}$ is compact, then $x \in \mathcal{C}$ is compact).

Exercise 1.8. Let I be a set and J_i be a filtered category for every $i \in I$, and let $f_i\colon J_i \to \mathcal{C}$ be functors. Construct the natural transformation $\mathrm{colim}_{\prod_i J_i} \prod_I \to \prod_I \mathrm{colim}_{J_i}$ of functors $\prod_I J_i \to \mathcal{C}$.

Exercise 1.9. Let $\mathcal{C}$ be a dualisable category

(1) Show that for $x \in \mathcal{C}$ the functor $(\mathcal{C}^\vee)^{\mathrm{op}} \to \mathrm{Sp}, y \mapsto \hom_{\mathcal{C}\otimes\mathcal{C}^\vee}(x \boxtimes y, \mathrm{coev}\,\mathbb{1})$ is corepresented by an object $x^\vee \in \mathcal{C}^\vee$, i.e. it is equivalent to $\hom_{\mathcal{C}^\vee}(-, x^\vee)$, where $\mathrm{coev}\colon \mathrm{Sp} \to \mathcal{C} \otimes \mathcal{C}^\vee$ denotes the coevaluation. This thus gives rise to a functor $(-)^\vee\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}^\vee$.

(2) Consider a ring R and $M \in \mathrm{Mod}_R$. Show that $M^\vee = \hom_R(M, R)$.

(3) Show that, under the equivalence $\mathcal{C}^\vee \simeq \mathrm{Fun}^L(\mathcal{C}, \mathrm{Sp})$, the object $x^\vee$ corresponds to the functor $\hom_{\mathrm{Ind}(\mathcal{C})}(Y(x), \widehat{Y}(-))$.

Exercise 1.10. Let $\mathcal{C}$ be a dualisable category and $\mathcal{A} \subseteq \mathcal{C}^\omega$ an idempotent complete stable subcategory.

(1) Show that the canonical map $\mathcal{C}^\omega/\mathcal{A} \to (\mathcal{C}/\mathrm{Ind}(\mathcal{A}))^\omega$ is an equivalence. **Hint:** *reduce to the case where $\mathcal{C}$ is compactly generated.*

(2) Deduce in particular that $(\mathcal{C}/\mathrm{Ind}(\mathcal{C}^\omega))^\omega \simeq 0$.

Exercise 1.11 (Thomason's theorem)**.** Let $\mathcal{C}$ be a stable category. Call a full stable subcategory $\mathcal{D} \subseteq \mathcal{C}$ dense if $\mathcal{D}$ generates $\mathcal{C}$ under retracts. Thomason's theorem states that for a dense stable subcategory the map $K_0(\mathcal{D}) \to K_0(\mathcal{C})$ is injective. Furthermore, the maps $(\mathcal{D} \subseteq \mathcal{C}) \mapsto (K_0(\mathcal{D}) \subseteq K_0(\mathcal{C}))$ and $H \subseteq K_0(\mathcal{C}) \mapsto \mathcal{C}^H = \{x \in \mathcal{C} : [x] \in H\}$ determine inverse equivalences between the collection of dense stable subcategories of $\mathcal{C}$ and subgroups of $K_0(\mathcal{C})$. Prove Thomason's theorem in the following steps:

(1) Show that $\mathcal{C}^H$ is a dense stable subcategory.

(2) Show that $H_{\mathcal{C}^H} = H$ where $H_{\mathcal{D}} := \mathrm{Im}(K_0(\mathcal{D}) \to K_0(\mathcal{C}))$.

(3) Show that $\mathcal{C}^{H_{\mathcal{D}}} = \mathcal{D}$. **Hint:** *Use Heller's criterion from Exercise 3.11. Alternatively, define an equivalence relation $\sim$ on $\pi_0(\mathcal{C}^\simeq)$ via $x \sim x'$ iff there are $d, d' \in \mathcal{D}$ with $x \oplus d \simeq x' \oplus d'$. Show that the map $K_0(\mathcal{C}) \to \pi_0(\mathcal{C}^\simeq)/\sim, [x] \mapsto [x]$ is a well defined group homomorphism with kernel $H_{\mathcal{D}}$.*

(4) Show that $K_0(\mathcal{D}) \to K_0(\mathcal{C})$ is injective, i.e. $K_0(\mathcal{D}) = H_{\mathcal{D}}$. **Hint:** *Apply the previous results for the category $\mathcal{D}$ and subgroup $N = \ker(K_0(\mathcal{D}) \to K_0(\mathcal{C})) \subseteq K_0(\mathcal{D})$.*

Exercise 1.12. Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ and $\mathcal{D} \subseteq \mathcal{C}$ be a full stable idempotent complete subcategory. Show that for $x \in \mathcal{C}/\mathcal{D}$ there is $y \in \mathcal{C}$ which gets send to $x \oplus x[1] \in \mathcal{C}/\mathcal{D}$ under the projection. In fact, show that for all $z \in \mathcal{C}/\mathcal{D}$ such that $[z] = 0 \in K_0(\mathcal{C}/\mathcal{D})$, there exists a lift of z to an object $\widetilde{z} \in \mathcal{C}$. **Hint:** *use Thomason's theorem from Exercise 1.11.*

Exercise 1.13. Let I be a possibly infinite set and $\mathcal{A}_i$ be a collection of small stable categories for all $i \in I$. Let $\mathcal{B}_i \subseteq \mathcal{A}_i$ be stable subcategories. Then show that the canonical map $\prod_I \mathcal{A}_i / \prod_I \mathcal{B}_i \to \prod_I (\mathcal{A}_i/\mathcal{B}_i)$ is an equivalence.

Exercise 1.14. Let p_i be the i–th prime number so that for example $p_1 = 2, p_2 = 3$, etc. Write $A_n := \mathbb{Z}[p_k^{-1}, k \geq n]$, so that for instance $A_1 = \mathbb{Q}$ and we have maps $A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow \cdots$. By restriction of scalars, we thus obtain a functor

$$\operatorname{colim}_n \mathcal{D}(A_n) \longrightarrow \mathcal{D}(\mathbb{Z}).$$

Show that this functor is not fully faithful.

2 Exercises from day 2

Exercise 2.1. Let $\mathcal{C}$ be a stable presentable category. Show the following facts:

(1) A map $f\colon x \to y$ in $\mathcal{C}$ is compact if and only for any filtered system $(z_i)_i$ in $\mathcal{C}$ together with a map $y \to \operatorname{colim}_i z_i$, the composite $x \to y \to \operatorname{colim}_i z_i$ factors through some z_j.

(2) id_x is compact in $\mathcal{C}$ if and only if x is compact.

(3) Compact maps in $\mathcal{C}$ form a two sided ideal.

(4) Suppose that $\mathcal{C}$ is compactly generated. Then a map in $\mathcal{C}$ is compact if and only if it factors through a compact object.

Exercise 2.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a colimit preserving functor between presentable stable categories.

(1) Suppose that $\mathcal{C}$ is dualisable. Show that F preserves compact morphisms if and only if F is strongly continuous.

(2) Suppose that $\mathcal{C}$ and $\mathcal{D}$ are dualisable. Show that F is strongly continuous if and only if the canonical transformation $\widehat{Y}_{\mathcal{D}} \circ F \to \operatorname{Ind}(F) \circ \widehat{Y}_{\mathcal{C}}$ is an equivalence.

Exercise 2.3 (Homological epimorphisms)**.** Consider a map $A \to B$ in $\operatorname{Alg}(\mathrm{Sp})$. Show that the map $B \otimes_A B \to B$ is an equivalence if and only if the restriction functor $\operatorname{Res}\colon \operatorname{Mod}_B \to \operatorname{Mod}_A$ is fully faithful.

Exercise 2.4. Suppose we have an inverse system of spectra $X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ such that $\pi_* X_i \leftarrow \pi_* X_{i+1}$ are the zero maps for all i. Show that $\lim_i X_i \simeq 0 \in \mathrm{Sp}$. Recall that this was used in Akhil's Lecture 2 in the proof of criterion (4) for dualisability in terms of compactly exhaustible maps. **Hint:** *use the Mittag–Leffler condition for vanishing of* $\lim_i^1$.

Exercise 2.5. Let $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^L_{\mathrm{st}}$ and recall the notations $\operatorname{Fun}^{\mathrm{acc}}(\mathcal{D}, \mathcal{E})$ and $\operatorname{Corr}(\mathcal{D}, \mathcal{E})$ from Sasha's Lecture 2.

(1) Work out the details of the equivalence $\operatorname{Fun}^{\mathrm{acc}}(\mathcal{D}, \mathcal{E}) \simeq \operatorname{Corr}(\mathcal{D}, \mathcal{E})$ as sketched in the lecture.

(2) Work out the details that the composition structures on $\operatorname{Fun}^{\mathrm{acc}}$ and Corr are compatible. That is, show that there is a naturally commuting square

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{acc}}(\mathcal{C},\mathcal{D}) \times \mathrm{Fun}^{\mathrm{acc}}(\mathcal{D},\mathcal{E}) & \xrightarrow{\circ} & \mathrm{Fun}^{\mathrm{acc}}(\mathcal{C},\mathcal{E}) \\ \downarrow \simeq & & \simeq \downarrow \\ \mathrm{Corr}(\mathcal{C},\mathcal{D}) \times \mathrm{Corr}(\mathcal{D},\mathcal{E}) & \xrightarrow{\circ} & \mathrm{Corr}(\mathcal{C},\mathcal{E}) \end{array}$$

for $\mathcal{C},\mathcal{D},\mathcal{E} \in \mathrm{Pr}^L_{\mathrm{st}}$.

Exercise 2.6. Let $\mathcal{A},\mathcal{B}$ be small stable categories. Recall that, for a category $\mathcal{C}$, we define $\mathrm{Pro}(\mathcal{C}) := \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. Show that there is an equivalence

$$\mathrm{Fun}^{\mathrm{acc,ex}}(\mathrm{Ind}\mathcal{A}, \mathrm{Ind}\mathcal{B}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{B}, \mathrm{Pro}(\mathrm{Ind}(\mathcal{A})))^{\mathrm{op}}.$$

3 Exercises from day 3

Exercise 3.1. Suppose that $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is a Karoubi sequence in $\mathrm{Cat}^{\mathrm{perf}}$. Show that $\mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B}) \to \mathrm{Ind}(\mathcal{C})$ is a short exact sequence in $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$.

Exercise 3.2. Let $\mathcal{C}$ be a dualisable category. For $a, b \in \mathcal{C}^{\omega_1}$, show that

$$\hom_{\mathrm{Calk}^{\mathrm{cont}}_{\omega_1}}(a,b) \simeq \hom_{\mathcal{C}}(a,b)/\hom_{\mathrm{Ind}(\mathcal{C})}(Y(a), \widehat{Y}(b))$$

by using that $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ is right adjoint to $\widehat{Y}$.

Exercise 3.3. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a strongly continuous functor between stable presentable categories. Show that if $\mathcal{C}$ is dualisable, then the localising subcategory of $\mathcal{A} \subseteq \mathcal{D}$ generated by the image of F is dualisable and the inclusion $\mathcal{A} \subseteq \mathcal{D}$ is strongly continuous.

Exercise 3.4. Consider a commutative square

$$\begin{array}{ccc} \mathcal{C}_0 & \longrightarrow & \mathcal{C}_1 \\ \downarrow F_0 & & \downarrow F_1 \\ \mathcal{D}_0 & \longrightarrow & \mathcal{D}_1 \end{array} \tag{$\square$}$$

in $\mathrm{Pr}^L_{\mathrm{st}}$. Show the following:

(1) If ($\square$) is a pullback square and F_1 is a localisation, then F_0 is a localisation and ($\square$) is also a pushout square.

(2) If ($\square$) is a pushout square and F_0 is fully faithful, then F_1 is fully faithful.

Exercise 3.5. Show that the forgetful functor $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}} \to \mathrm{Pr}^L_{\mathrm{st}}$ preserves the following types of limits:

(1) finite products;

(2) fibers of strongly continuous localisations;

(3) pullbacks where one leg is a strongly continuous localisation.

Exercise 3.6. Let $(\mathcal{C}_i)_i$ be a family of stable presentable categories. Show that if each $\mathcal{C}_i$ is compactly generated, then $\prod_i \mathcal{C}_i$ is compactly generated and $(\prod_i \mathcal{C}_i)^\omega \simeq \bigoplus_i \mathcal{C}_i^\omega$, where the coproduct is formed in $\mathrm{Cat}^{\mathrm{perf}}$.

Exercise 3.7 (Generalisation of Tamme's excision theorem)**.** Consider a pullback of the form (□) in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ and assume that F_1 is a localisation. Show that for any localising invariant $E\colon \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathcal{E}$, the square

$$\begin{array}{ccc} E(\mathcal{C}_0) & \longrightarrow & E(\mathcal{C}_1) \\ \downarrow & & \downarrow \\ E(\mathcal{D}_0) & \longrightarrow & E(\mathcal{D}_1) \end{array}$$

is a pullback square.

Exercise 3.8. Work out the details that we have a Bousfield localisation

$$\mathrm{Fun}(\mathbb{Q}_{\leq}^{\mathrm{op}}, \mathrm{Sp}) \underset{\phi^R}{\overset{\phi}{\rightleftarrows}} \prod_{\mathbb{Q}} \mathrm{Sp}$$

given by $\phi(F)_a := \mathrm{cofib}(\mathrm{colim}_{b>a} F(b) \to F(a))$ and $\phi^R((X_a)_{a\in\mathbb{Q}})(b) = X_b$. This was used in Sasha's Lecture 2 to obtain a short exact sequence in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ with kernel $\mathrm{Shv}_{\geq 0}(\mathbb{R}, \mathrm{Sp})$.

Exercise 3.9 (Waldhausen's addtivity trick)**.** Let $F\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ be a localising invariant. Show that for any fiber sequence $f \to g \to h$ in $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{B})$ there is an equivalence $F(g) \simeq F(f) \oplus F(h)$ in $\hom_{\mathcal{E}}(F(\mathcal{A}), F(\mathcal{B}))$. **Hint:** *Use the split Karoubi sequence $\mathcal{C} \to \mathcal{C}^{\Delta^1} \to \mathcal{C}$ to first construct a splitting $F(\mathcal{C}^{\Delta^1}) \simeq F(\mathcal{C}) \oplus F(\mathcal{C})$.*

Exercise 3.10 (Universal K-equivalences)**.** An exact functor $f\colon \mathcal{A} \to \mathcal{B}$ between small stable categories is called a *universal K-equivalence* if there is an exact functor $g\colon \mathcal{B} \to \mathcal{A}$ such that $[gf] = [\mathrm{id}_{\mathcal{A}}]$ in $K_0(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}, \mathcal{A}))$ and $[fg] = [\mathrm{id}_{\mathcal{B}}]$ in $K_0(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{B}, \mathcal{B}))$.

Show that f is a univeral K-equivalence if and only if for every additive invariant $F\colon \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$, the map $F(f)\colon F(\mathcal{A}) \to F(\mathcal{B})$ is an equivalence. (If you don't know what additive invariants are, just prove the $\implies$ direction for any localising invariant)

Exercise 3.11 (Heller's criterion)**.** Let $\mathcal{C}$ be a small stable category. Show that for $x, y \in \mathcal{C}$ the following are equivalent:

(1) $[x] = [y]$ in $K_0(\mathcal{C})$.

(2) There exist $u, v, z \in \mathcal{C}$ such and fiber sequences $u \to x \oplus z \to v$ and $u \to y \oplus z \to v$.

Hint: *Define an equivalence relation $\sim$ on $\pi_0(\mathcal{C}^{\simeq})$ by (2) and construct an equivalence $K_0(\mathcal{C}) \simeq \pi_0(\mathcal{C}^{\simeq})/\sim$.*

As an application, show that for a family $(\mathcal{C}_i)_{i\in I}$ of small stable categories, the natural map $K_0(\prod_i \mathcal{C}_i) \to \prod_i K_0(\mathcal{C}_i)$ is an equivalence.